Maximum Flow Problem

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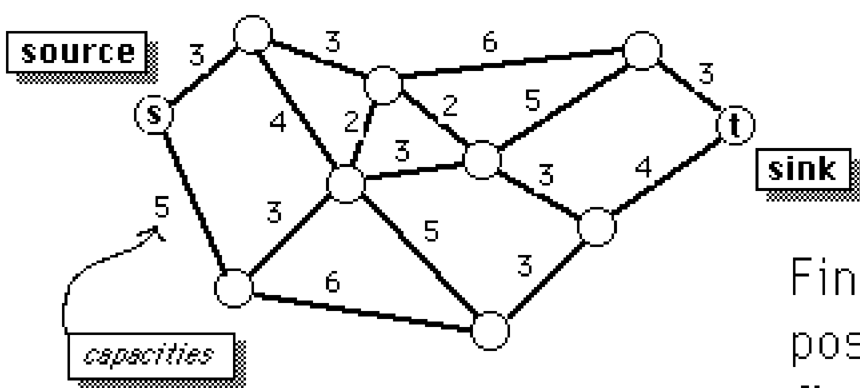


Find the maximum possible amount of flow in the network from the source **s** to the sink **t**

ALGORITHM

Given: a network with designated source & sink, each arc having a capacity in each direction. (Capacity of arc (i,j) need not equal that of (j,i))

Step 0 Initially, let the flow in each arc be zero.

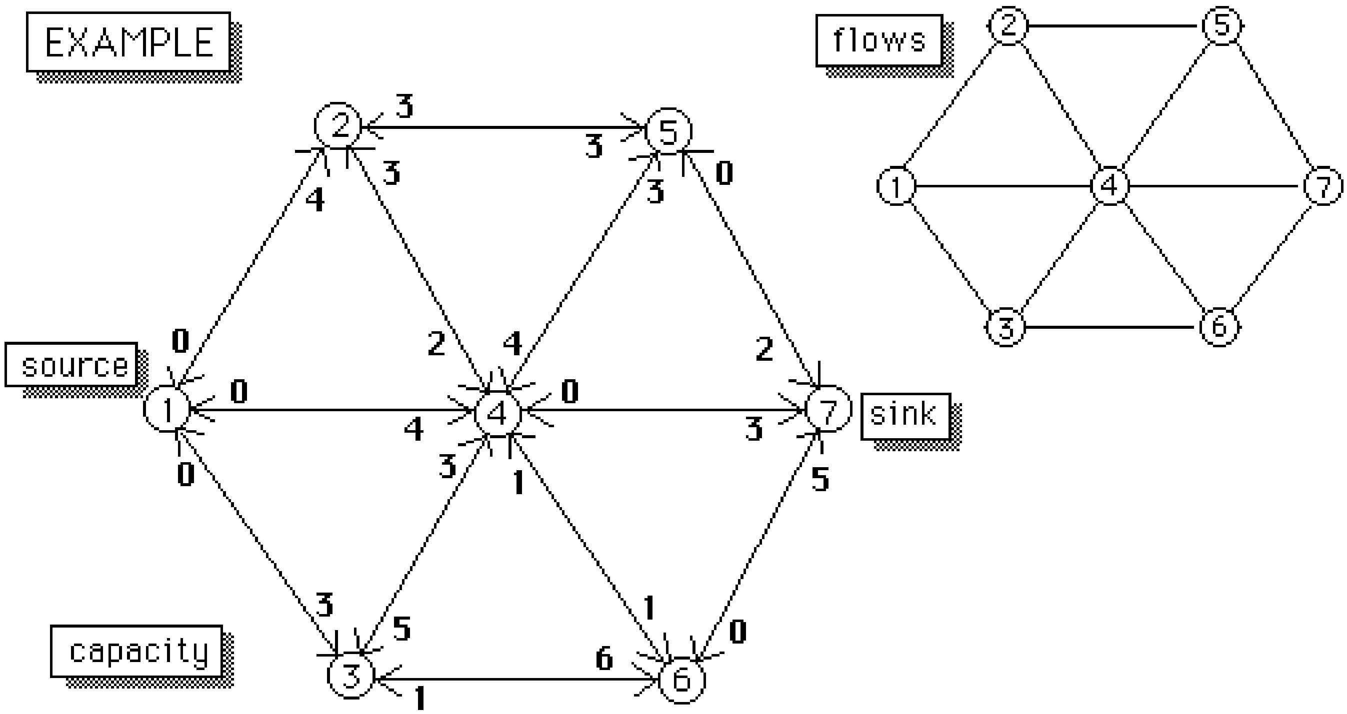
Step 1 Find any path from source to sink that has positive flow capacity (in direction of flow) for every arc in the path. If no such path exists, STOP.

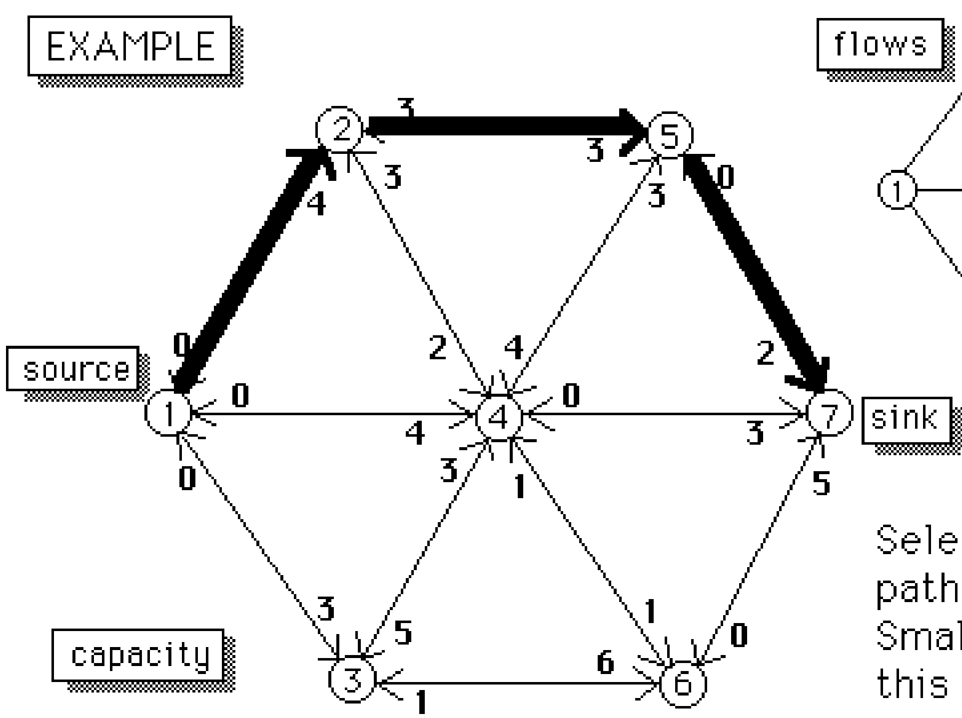
(For example, try to construct a spanning tree, using only arcs with positive capacity.)

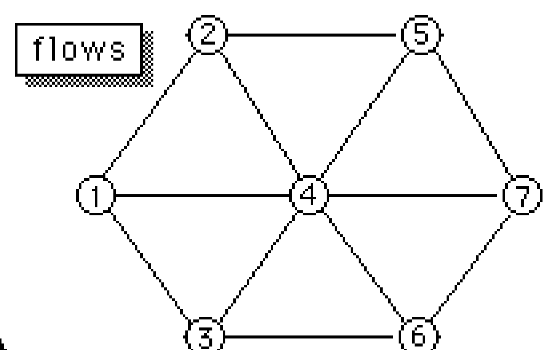
Step 2 Find the smallest arc capacity **k** on this path *(the flow-augmenting path)*. Increase the flow in this path by **k**.

Step 3 For each arc in the flow-augmenting path, **reduce** all capacities in the direction of the flow by the amount **k**, and **increase** all capacities in the direction opposite the flow by **k**.

Return to Step 1.

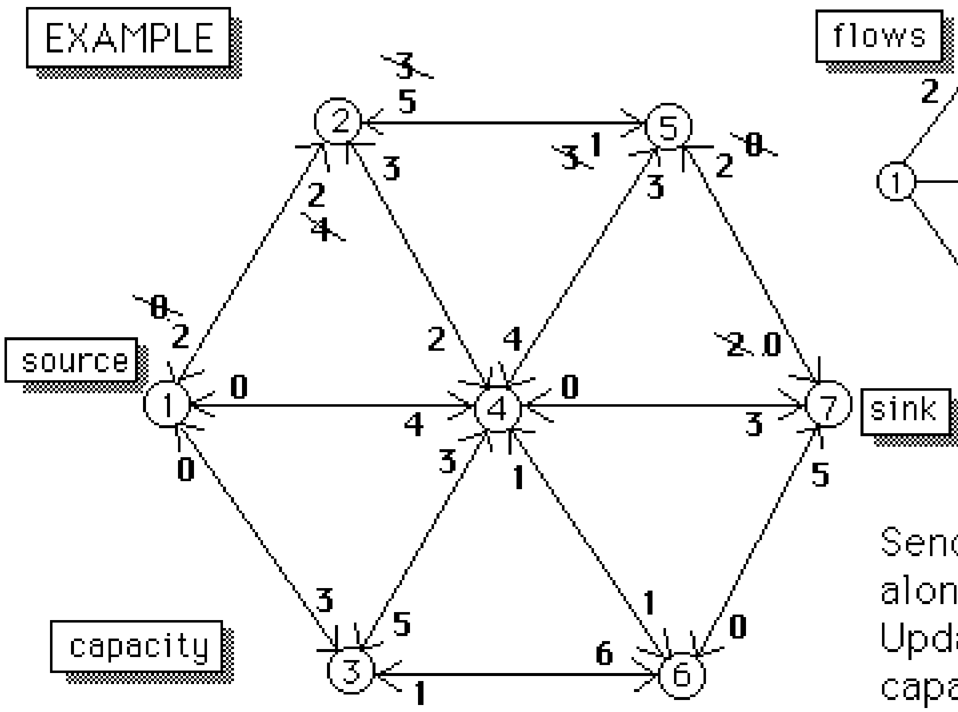


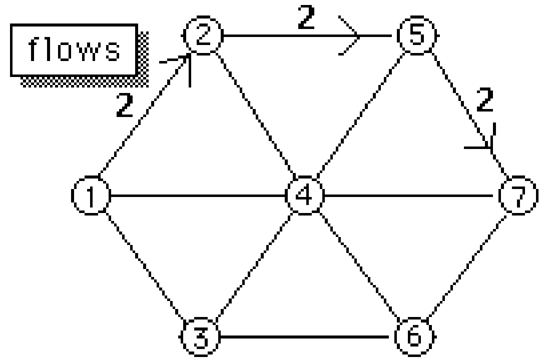




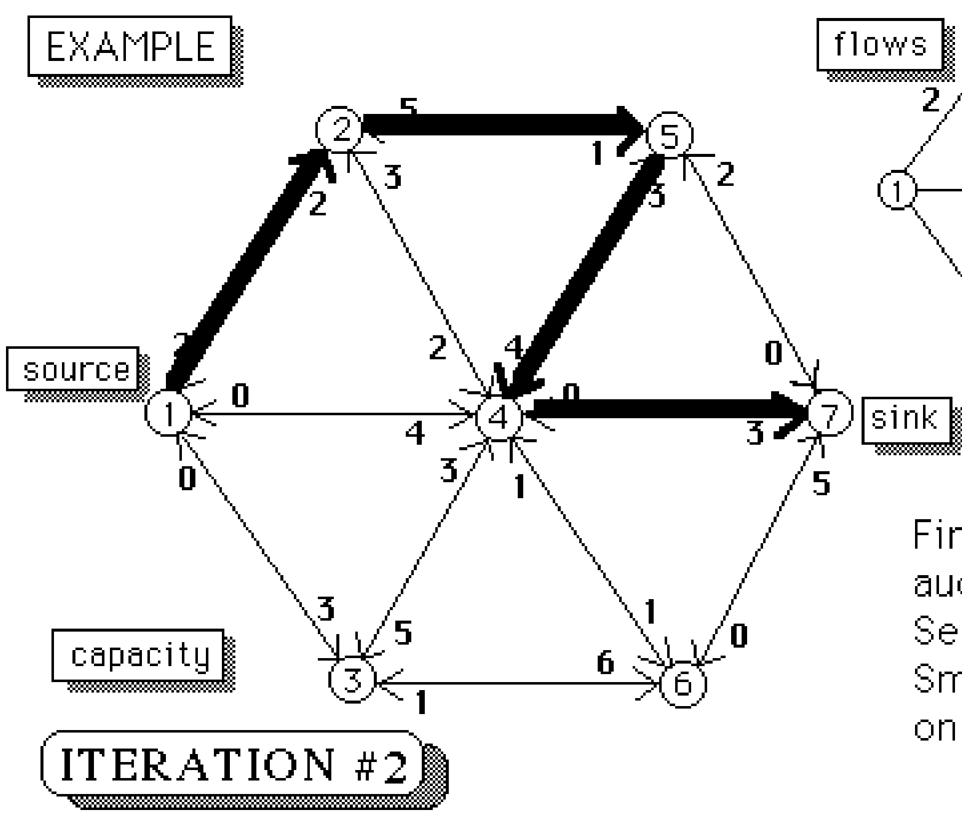
Select flow-augmenting path 1-2-5-7.
Smallest capacity on this path is 2.

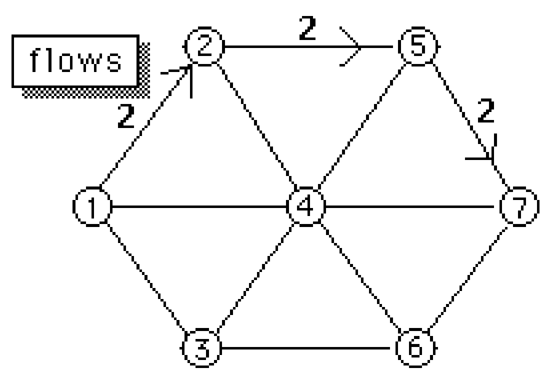
ITERATION #1



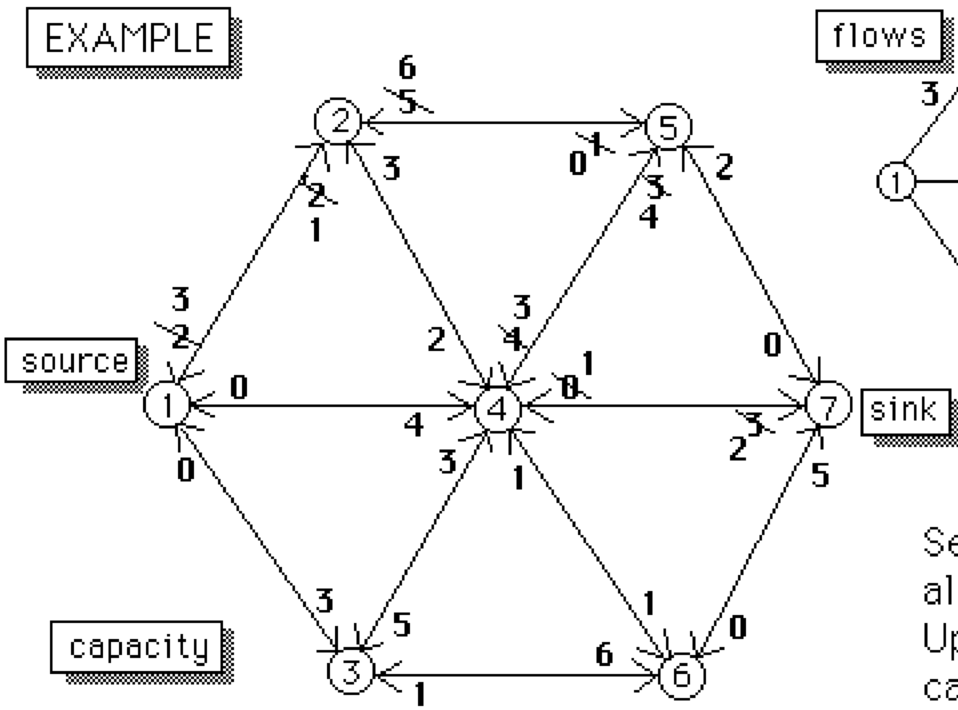


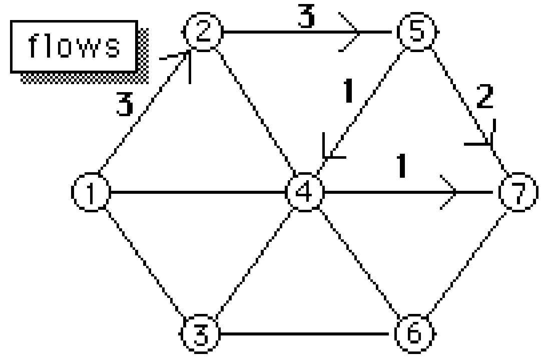
Send 2 units of flow along the path.
Update the arc capacities (forward & backward).



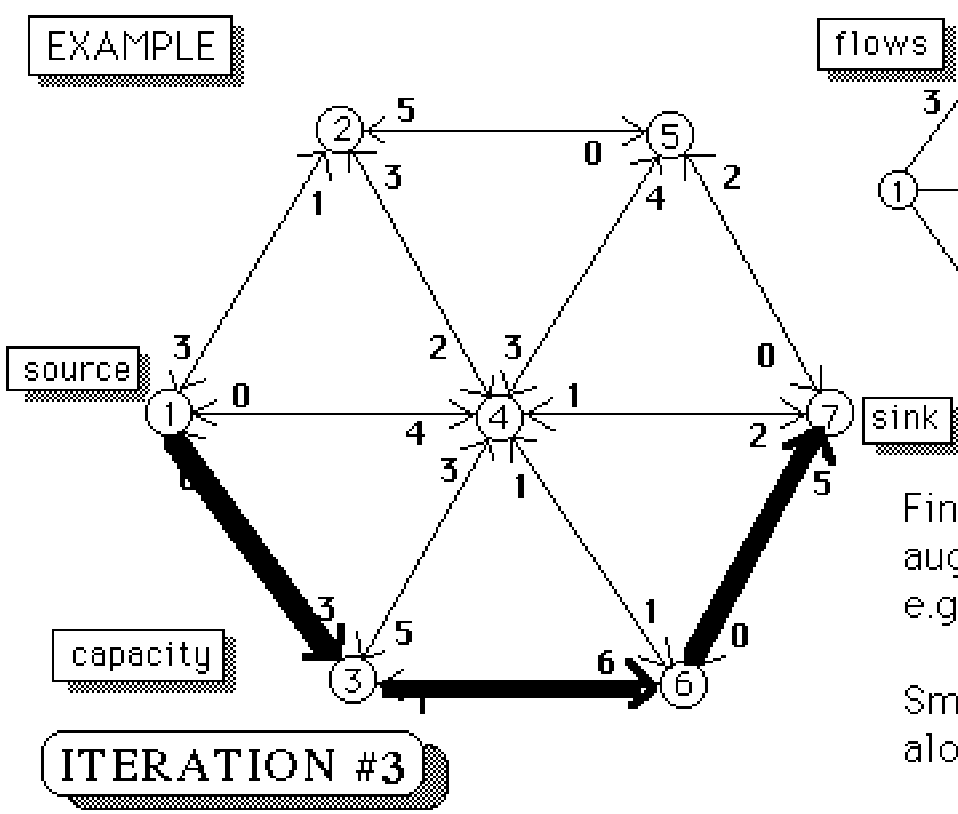


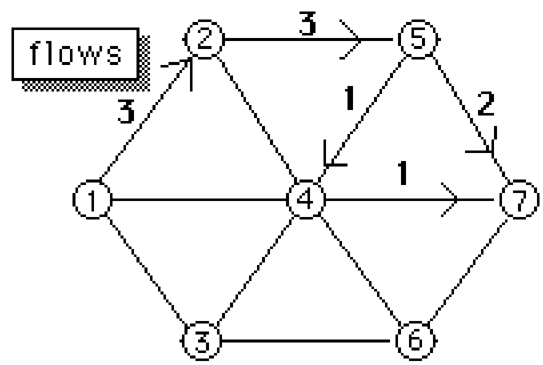
Find the next flow-augmenting path.
Select 1-2-5-4-7.
Smallest capacity on path is 1.





Send 1 unit of flow along the path. Update the arc capacities (forward & backward) along the path.

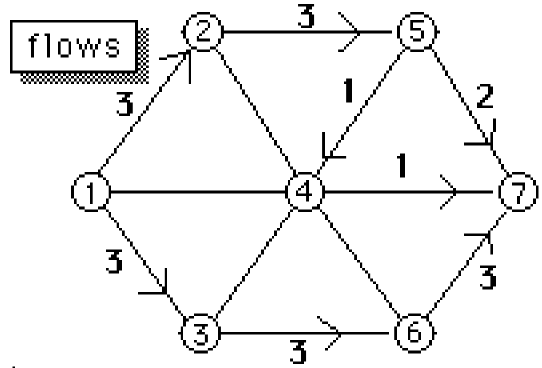


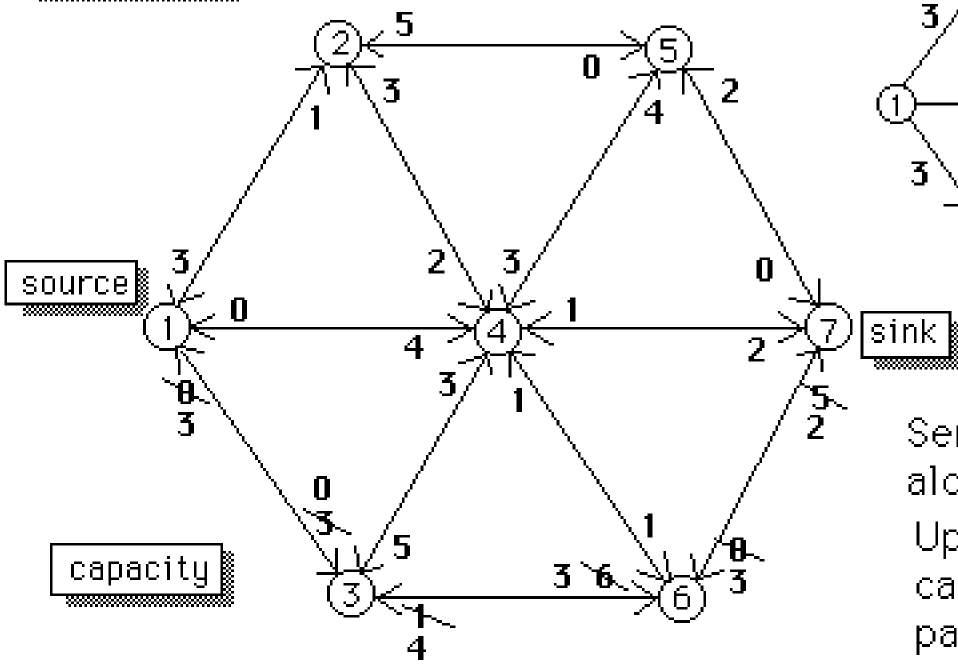


Find another flow-augmenting path, e.g. 1-3-6-7

Smallest capacity along path is 3.

EXAMPLE

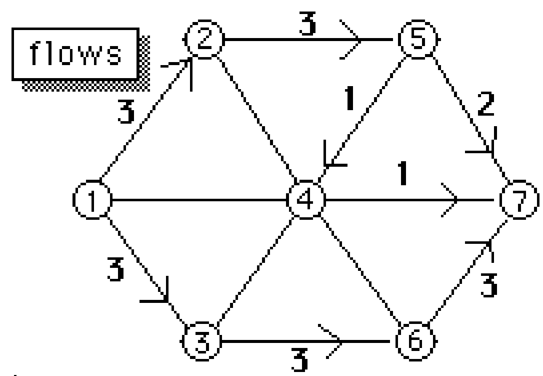


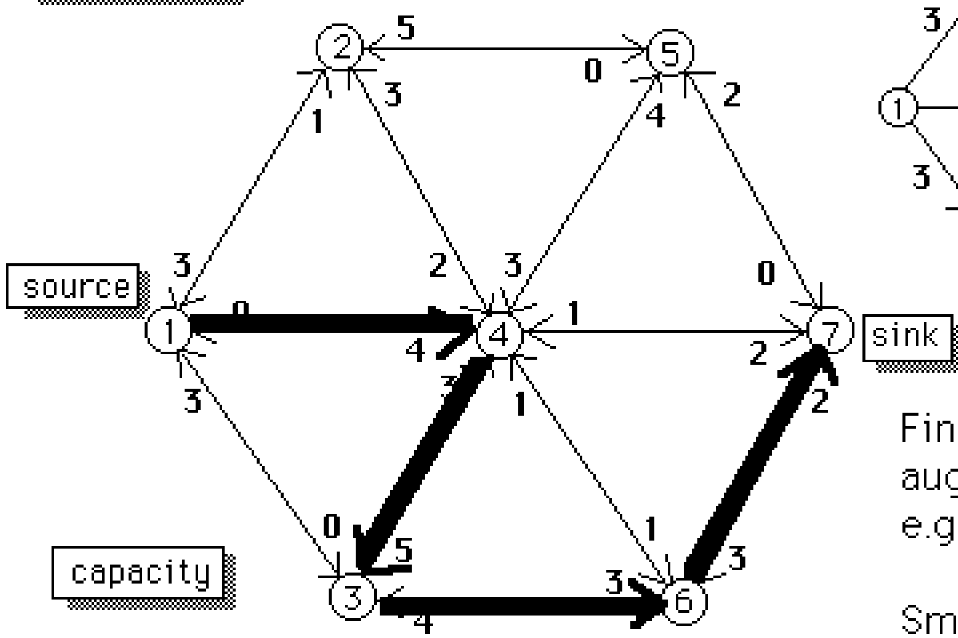


Send 3 units of flow along the path

Update the capacities along the path (forward & backward.)

EXAMPLE

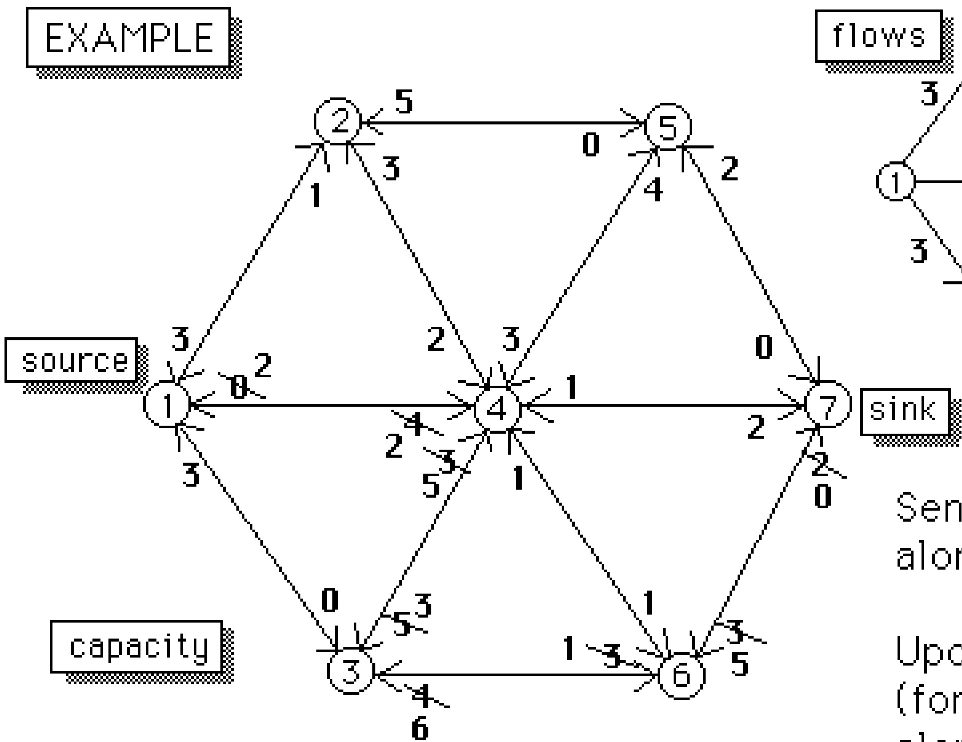


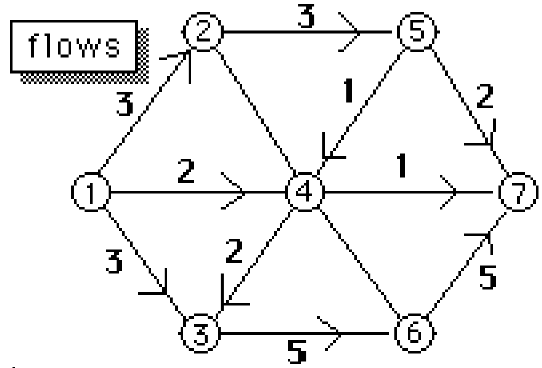


Find another flow-augmenting path, e.g., 1-4-3-6-7.

Smallest capacity along the path is 2.

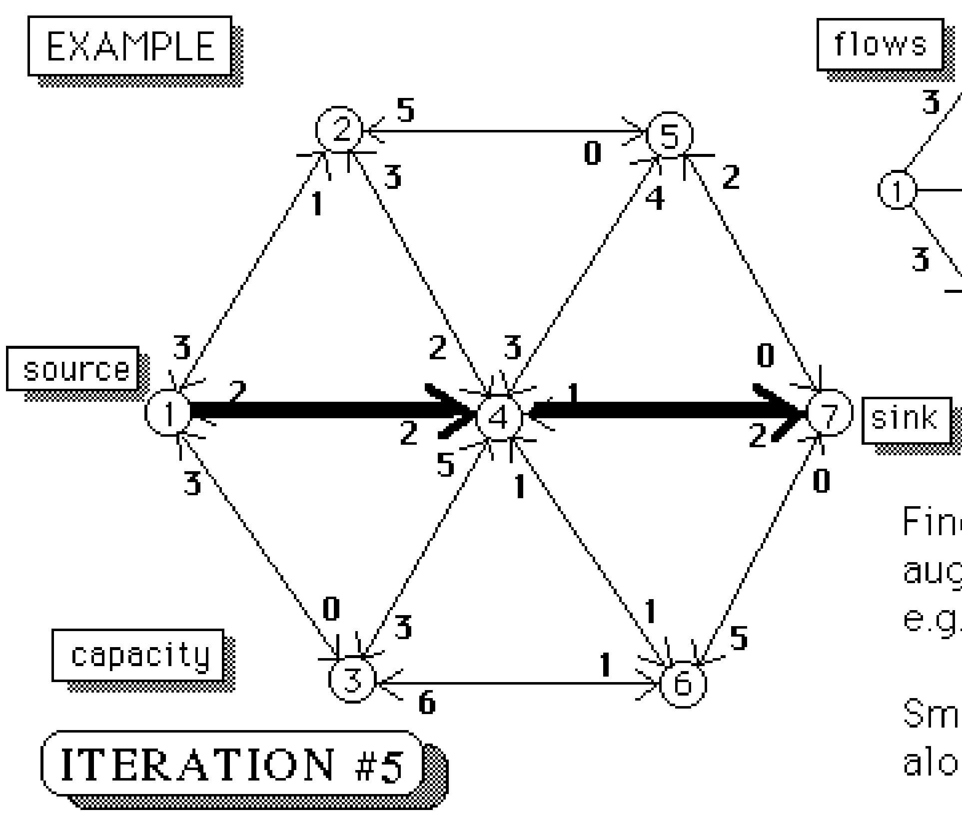
ITERATION #4

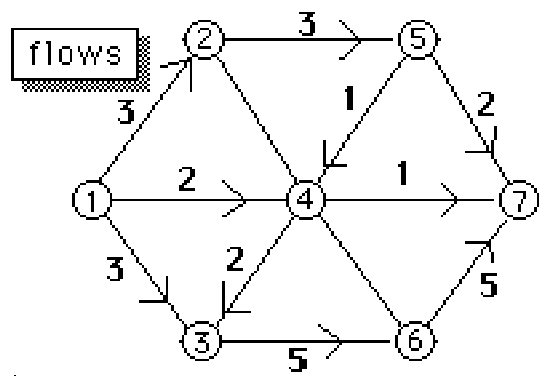




Send 2 units of flow along the path.

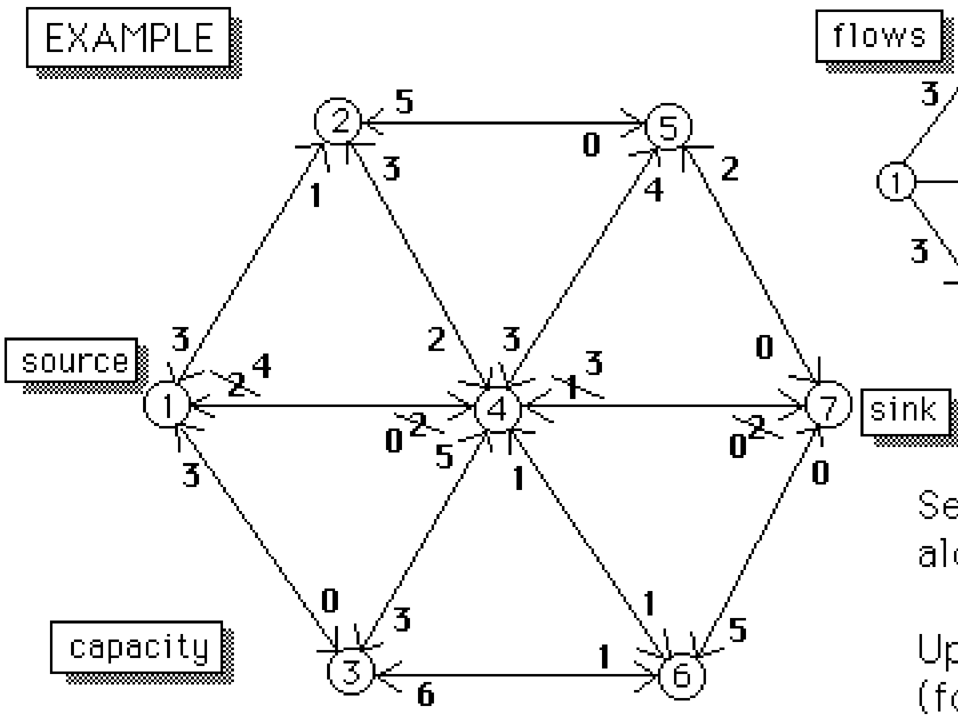
Update the capacities (forward & backward) along the path.

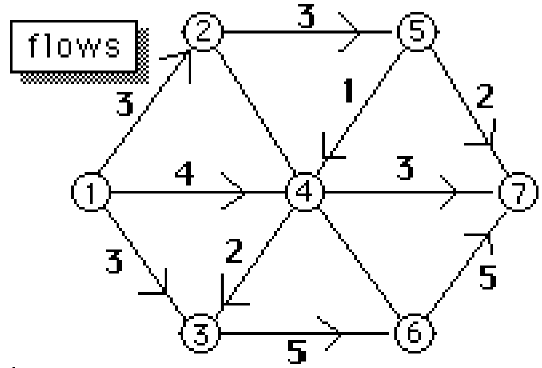




Find the next flow-augmenting path, e.g., 1-4-7.

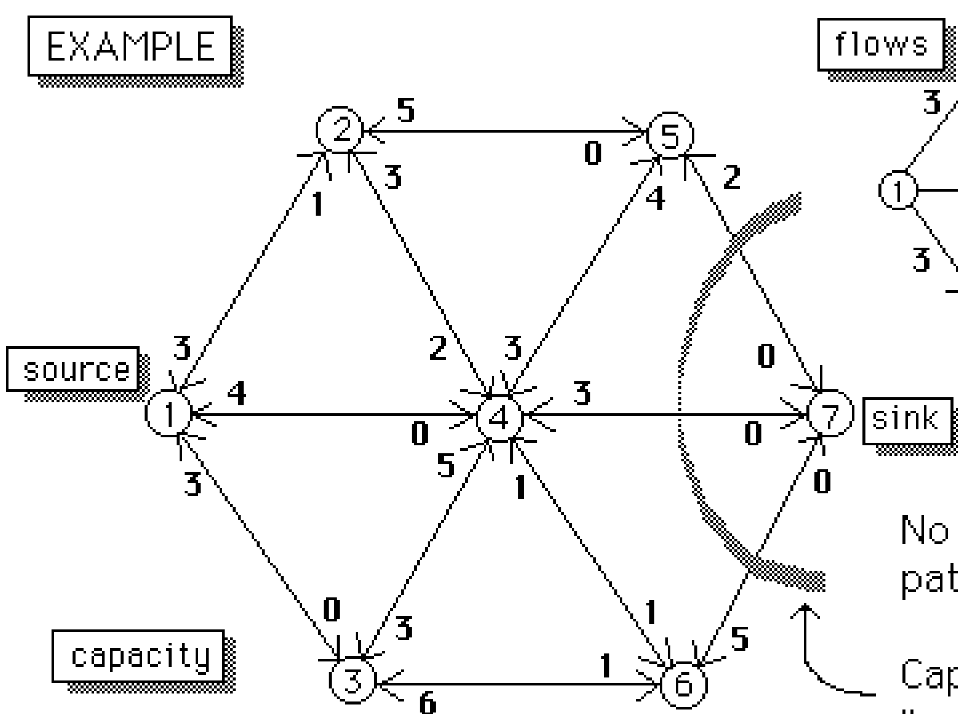
Smallest capacity along this path is 2.

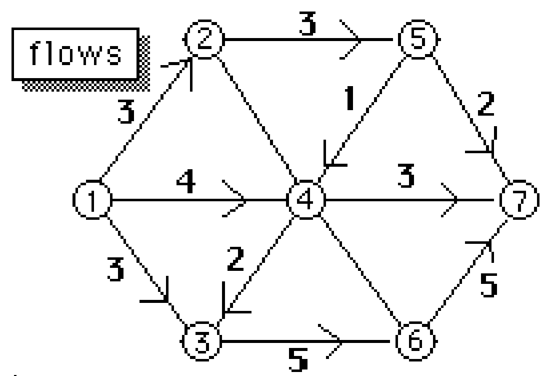




Send 2 units of flow along this path.

Update the capacities (forward & backward).





No flow-augmenting path can now be found.

Capacity across this "cut" is zero!

ITERATION #6

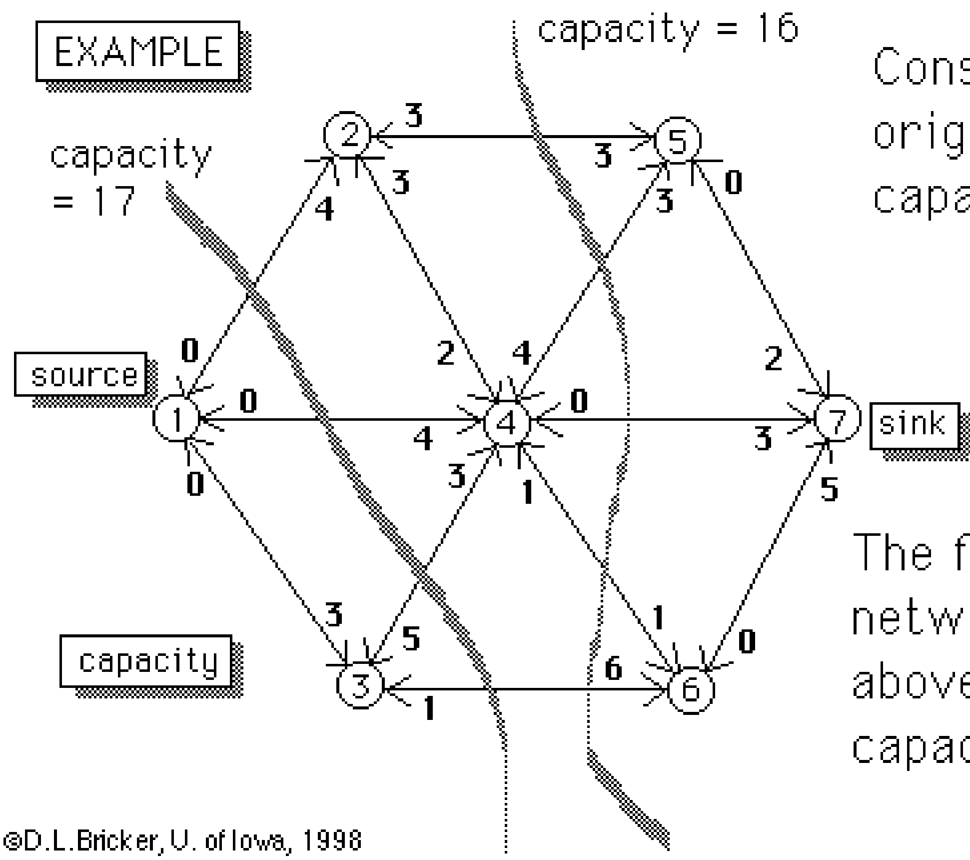
Definition

A *cut* of a network is a partition of the node set N into 2 subsets, N_1 and N_2 , such that

- $N = N_1 \cup N_2$,
- $N_1 \cap N_2 = \emptyset$,
- the source node is in N_1,
- the sink node is in N_2

The *capacity* of the cut is $\sum_{i \in N_1} \sum_{j \in N_2} c_{ij}$

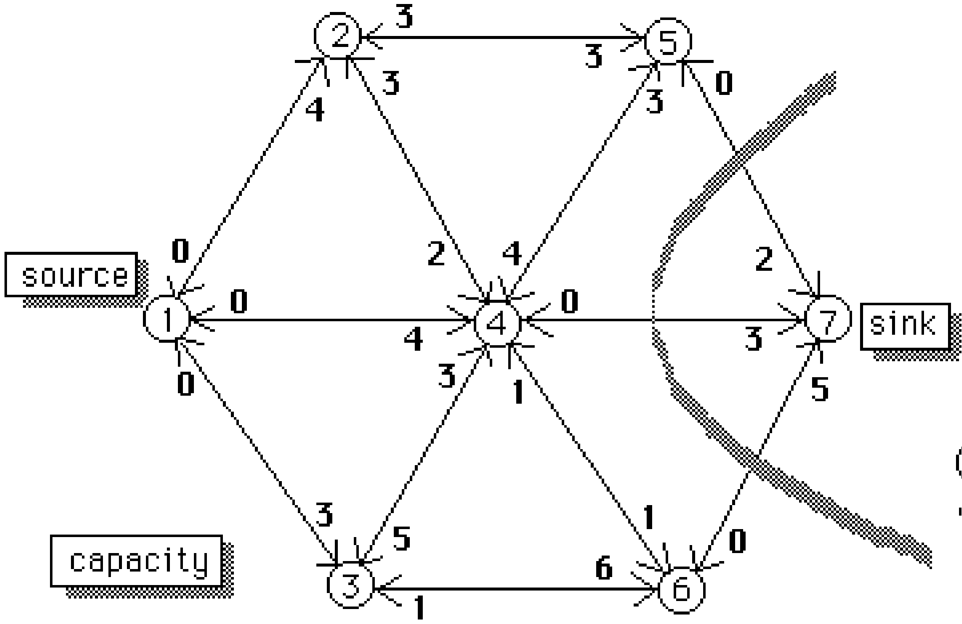
EXAMPLE



Consider the original arc capacities

The flow in a network is bounded above by the capacity of any cut.

EXAMPLE



Capacity of this "cut" is 10

= maximum flow

MAX-FLOW/MIN-CUT THEOREM

The maximum flow in a network is equal to the capacity of the cut having the minimum cut capacity.