# Machine Replacement Problem

# Markov Decision Model

This Hypercard stack was prepared by:
Dennis L. Bricker,
Dept. of Industrial Engineering,
University of Iowa,
Iowa City, Iowa 52242
e-mail: dbricker@icaen.uiowa.edu

author

## Machine Replacement Problem

At the beginning of each month, a machine is inspected and classified as:

1) Good as new
2) Operable, with minor deterioration
3) Operable, with major deterioration
4) Inoperable

## Machine Replacement Problem

After determining the state of the machine, a decision must be made:

1) Keep the machine another month
2) Replace the machine with a new machine



## Machine Replacement Problem

A replacement machine costs $3000, minus trade-in value:

$1000 if in state 2
500 if in state 3
0 if in state 4

Monthly operating costs are $100, $200, and $500 for a machine in states 1, 2, & 3, respectively.

# Machine Replacement Problem

## Survival probabilities

| from: \ to: State | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 0.75 | 0.1875 | 0.0625 | 0 |
| 2 | — | 0.75 | 0.1875 | 0.0625 |
| 3 | — | — | 0.75 | 0.25 |

*What is the optimal replacement policy?*



States

| i | name |
|---|---|
| 1 | Good as new |
| 2 | Minor deterioration |
| 3 | Major deterioration |
| 4 | Failed |

Actions

| k | name |
|---|---|
| 1 | Keep |
| 2 | Replace |

## Cost Matrix

| k | name | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 1 | Keep | 100 | 200 | 500 | 9999 |
| 2 | Replace | 9999 | 2100 | 2600 | 3100 |

(Rows ~ actions, Columns ~ states)

A value of 9999 above signals an infeasible action in a state.

*(includes cost of operating the new machine, if decision is to replace)*



## Transition Probabilities

**Action: Keep**

| from \ to | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 0.75 | 0.1875 | 0.0625 | 0 |
| 2 | 0 | 0.75 | 0.1875 | 0.0625 |
| 3 | 0 | 0 | 0.75 | 0.25 |
| 4 | 0 | 0 | 0 | 1 |

**Action: Replace**

| from \ to | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 0.75 | 0.1875 | 0.0625 | 0 |
| 2 | 0.75 | 0.1875 | 0.0625 | 0 |
| 3 | 0.75 | 0.1875 | 0.0625 | 0 |
| 4 | 0.75 | 0.1875 | 0.0625 | 0 |

*Rows are identical, since each month begins with a new machine, regardless of current condition*

☞ Linear Programming Approach

☞ Policy Iteration Method



# Linear Programming Model

*What is the policy which minimizes the average cost/month in steady state?*

$$\text{Minimize} \sum_{i=1}^{N} C_i^k X_i^k$$

*where*

$X_i^k$ = probability that machine is in state #i and decision #k is selected

## LP Tableau

| k: | 1 | 1 | 2 | 1 | 2 | 2 | RHS |
|---|---|---|---|---|---|---|---|
| i: | 1 | 2 | 2 | 3 | 3 | 4 | |
| Min | 100 | 200 | 2100 | 500 | 2600 | 3100 | |
| | 0.25 | 0 | ¯0.75 | 0 | ¯0.75 | ¯0.75 | 0 |
| | ¯0.1875 | 0.25 | 0.8125 | 0 | ¯0.1875 | ¯0.1875 | 0 |
| | ¯0.0625 | ¯0.1875 | ¯0.0625 | 0.25 | 0.9375 | ¯0.0625 | 0 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

$X_i^k$ = probability that machine is in state #i and decision #k is selected



Iteration 0

*Initial policy: keep until the machine fails*

*basic:* ★★ (columns k=1,i=1; k=1,i=2), ★ (k=1,i=3), ★ (k=2,i=4)

| k: | 1 | 1 | 2 | 1 | 2 | 2 | |
|---|---|---|---|---|---|---|---|
| i: | 1 | 2 | 2 | 3 | 3 | 4 | rhs |
| Min | 0 | 0 | 541.463 | 0 | ¯304.878 | 0 | ¯548.78 |
| | 1 | 0 | ¯2.04878 | 0 | ¯1.17073 | 0 | 0.292683 |
| | 0 | 1 | 1.95122 | 0 | ¯1.17073 | 0 | 0.292683 |
| | 0 | 0 | 0.780488 | 1 | 2.73171 | 0 | 0.317073 |
| | 0 | 0 | 0.317073 | 0 | 0.609756 | 1 | 0.097561 |

i~state, k~action

*Initial basis is obtained by selecting one basic variable for each state.*

*Steadystate distribution resulting from this policy*

Iteration 0

Policy: (Cost= 548.78 )

| State | Action | P{i} |
|---|---|---|
| 1 Good as new | 1 Keep | 0.292683 |
| 2 Minor deterioration | 1 Keep | 0.292683 |
| 3 Major deterioration | 1 Keep | 0.317073 |
| 4 Failed | 2 Replace | 0.097561 |



*Choose a column having negative reduced cost, and enter it into the basis:*

| basic | ★ | ★ | | ★ | | ★ | |
|---|---|---|---|---|---|---|---|
| k: | 1 | 1 | 2 | 1 | 2 | 2 | |
| i: | 1 | 2 | 2 | 3 | 3 | 4 | rhs |
| Min | 0 | 0 | 541.463 | 0 | ⁻304.878 | 0 | ⁻548.78 |
| | 1 | 0 | ⁻2.04878 | 0 | ⁻1.17073 | 0 | 0.292683 |
| | 0 | 1 | 1.95122 | 0 | ⁻1.17073 | 0 | 0.292683 |
| | 0 | 0 | 0.780488 | 1 | 2.73171 | 0 | 0.317073 |
| | 0 | 0 | 0.317073 | 0 | 0.609756 | 1 | 0.097561 |

*choose pivot row, using minimum ratio test*

$$\text{minimum}\left\{\frac{0.31707}{2.7317}, \frac{0.09756}{0.60975}\right\}$$

$$= \frac{0.31707}{2.7317} = 0.116071$$

## Iteration 1

| | ★ | ★ | | | ★ | ★ | |
|---|---|---|---|---|---|---|---|
| k: | 1 | 1 | 2 | 1 | 2 | 2 | |
| i: | 1 | 2 | 2 | 3 | 3 | 4 | rhs |
| Min | 0 | 0 | 628.571 | 111.607 | 0 | 0 | ¯513.393 |
| | 1 | 0 | ¯1.71429 | 0.428571 | 0 | 0 | 0.428571 |
| | 0 | 1 | 2.28571 | 0.428571 | 0 | 0 | 0.428571 |
| | 0 | 0 | 0.285714 | 0.366071 | 1 | 0 | 0.116071 |
| | 0 | 0 | 0.142857 | ¯0.223214 | 0 | 1 | 0.0267857 |

i~state, k~action

$X_3^2$ has replaced $X_3^1$ in the basis



## Iteration 1

Policy: (Cost= 513.393 )

| State | Action | P{i} |
|---|---|---|
| 1 Good as new | 1 Keep | 0.428571 |
| 2 Minor deterioration | 1 Keep | 0.428571 |
| 3 Major deterioration | 2 Replace | 0.116071 |
| 4 Failed | 2 Replace | 0.0267857 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | ★ | ★ | | | ★ | ★ | |
| k: | 1 | 1 | 2 | 1 | 2 | 2 | |
| i: | 1 | 2 | 2 | 3 | 3 | 4 | rhs |
| Min | 0 | 0 | 628.571 | 111.607 | 0 | 0 | ¯513.393 |
| | 1 | 0 | ¯1.71429 | 0.428571 | 0 | 0 | 0.428571 |
| | 0 | 1 | 2.28571 | 0.428571 | 0 | 0 | 0.428571 |
| | 0 | 0 | 0.285714 | 0.366071 | 1 | 0 | 0.116071 |
| | 0 | 0 | 0.142857 | ¯0.223214 | 0 | 1 | 0.0267857 |

i~state, k~action

*Reduced costs are nonnegative...*
*the optimality condition is satisfied!*



**Optimal Policy**

| State | Action |
|---|---|
| 1 Good as new | 1 Keep |
| 2 Minor deterioration | 1 Keep |
| 3 Major deterioration | 2 Replace |
| 4 Failed | 2 Replace |

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# Policy Iteration Method

- ☞ Average cost per month
- ☞ Present value of all future costs

⏎



## Machine Replacement Example

| State | Action |
|---|---|
| 1 Good as new | 1 Keep |
| 2 Minor deterioration | 1 Keep |
| 3 Major deterioration | 1 Keep |
| 4 Failed | 2 Replace |

g(R) = 548.78

| i | Vi |
|---|---|
| 1 | ¯3000 |
| 2 | ¯1541.46 |
| 3 | ¯195.122 |
| 4 | 0 |

⏎

## Machine Replacement Example

Policy Improvement Step: Evaluation of alternate actions

### State #2, Minor deterioration

Current Policy: action #1, Keep

g(R)+Vi(R) = ¯992.683

| k | name | C' | ΔC |
|---|---|---|---|
| 1 | Keep | ¯992.683 | 0 |
| 2 | Replace | ¯451.22 | 541.463 |

*no improvement can be achieved by changing action in this state.*

C'[k] = cost if action k is selected for one stage
ΔC[k] = improvement (if <0)



## Machine Replacement Example

Policy Improvement Step: Evaluation of alternate actions

### State #3, Major deterioration

Current Policy: action #1, Keep

g(R)+Vi(R) = 353.659

| k | name | C' | ΔC |
|---|---|---|---|
| 1 | Keep | 353.659 | 0 |
| 2 | Replace | 48.7805 | ¯304.878 |

*improvement*

C'[k] = cost if action k is selected for one stage
ΔC[k] = improvement (if <0)

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## Machine Replacement Example

| State | Action |
|---|---|
| 1 Good as new | 1 Keep |
| 2 Minor deterioration | 1 Keep |
| 3 Major deterioration | 2 Replace |
| 4 Failed | 2 Replace |

*new policy*

### *Value Determination*

g(R) = 513.393

| i | Vi |
|---|---|
| 1 | ¯3000 |
| 2 | ¯1628.57 |
| 3 | ¯500 |
| 4 | 0 |



## Machine Replacement Example

Policy Improvement Step: Evaluation of alternate actions

### State #2, Minor deterioration

Current Policy: action #1, Keep

g(R)+Vi(R) = ¯1115.18

| k | name | C' | ΔC |
|---|---|---|---|
| 1 | Keep | ¯1115.18 | 0 |
| 2 | Replace | ¯486.607 | 628.571 |

*no improvement can be achieved by changing action in this state.*

C'[k] = cost if action k is selected for one stage
ΔC[k] = improvement (if <0)

## Machine Replacement Example

Policy Improvement Step: Evaluation of alternate actions

**State #3, Major deterioration**

Current Policy: action #2, Replace

g(R)+Vi(R) = 13.3929

| k | name | C' | ΔC |
|---|---|---|---|
| 1 | Keep | 125 | 111.607 |
| 2 | Replace | 13.3929 | 0 |

*no improvement can be achieved by changing action in this state.*

C'[k] = cost if action k is selected for one stage
ΔC[k] = improvement (if <0)

***The current policy is optimal!***



Minimizing:

Present value of all future costs

i.e., MDP with discounting is used.

Assume a rate of return of 1.5% per month (18% per year)

$$\beta = \frac{1}{1+r} = \frac{1}{1.015} = 0.985222$$

That is, the present value of a $1 cost next month is $0.985222

*Let's begin with an initial policy:*
*keep machine until it fails*
*i.e., R = (1, 1, 1, 2)*

| | State | Action |
|---|---|---|
| 1 | Good as new | 1 Keep |
| 2 | Minor deterioration | 1 Keep |
| 3 | Major deterioration | 1 Keep |
| 4 | Failed | 2 Replace |

Discount factor = 0.985222
(rate of return = 1.5%)



## Value Determination

Solve the system of equations:

$$v_i(R) = C_i^{k_i} + \beta \sum_{j \in S} p_{ij}^{k_i} v_j(R) \quad \forall\ i \in S$$

$$P^R = \begin{bmatrix} 0.75 & 0.1875 & 0.0625 & 0 \\ 0 & 0.75 & 0.1875 & 0.0625 \\ 0 & 0 & 0.75 & 0.25 \\ 0.75 & 0.1875 & 0.0625 & 0 \end{bmatrix}$$

$$v_i(R) = C_i^{k_i} + \beta \sum_{j\in S} p_{ij}^{k_i} v_j(R) \quad \forall\ i \in S$$



$$v_i(R) = C_i^{k_i} + \beta \sum_{j\in S} p_{ij}^{k_i} v_j(R) \quad \forall\ i \in S$$

$$\begin{cases} v_1 = 100 + 0.98522\,(0.75v_1 + 0.1875v_2 + 0.0625v_3) \\ v_2 = 200 + 0.98522\,(0.75v_2 + 0.1875v_3 + 0.0625v_4) \\ v_3 = 500 + 0.98522\,(0.75v_3 + 0.25v_4) \\ v_4 = 3100 + 0.98522\,(0.75v_1 + 0.1875v_2 + 0.0625v_3) \end{cases}$$

Solution:

| i | $v_i$ |
|---|---|
| 1 | 35567.4 |
| 2 | 36960.8 |
| 3 | 38299.4 |
| 4 | 38567.4 |

| i | $v_i$ |
|---|---|
| 1 | 35567.4 |
| 2 | 36960.8 |
| 3 | 38299.4 |
| 4 | 38567.4 |

That is, $35,567.40 invested at 1.5% per month interest would sufficient to pay all future operation and replacement cost for the machine, if it is initially in state 1, i.e., "good as new"



# Policy Improvement

Policy Improvement Step: Evaluation of alternate actions

State #2, Minor deterioration

Current Policy: action #1, Keep

Evaluate alternative action: #2, Replace

$$v_i' = C_i^{k'_i} + \beta \sum_j p_{ij}^{k'_i} v_j \quad i=2,\ k_i'=2$$

| i | $v_i$ |
|---|---|
| 1 | 35567.4 |
| 2 | 36960.8 |
| 3 | 38299.4 |
| 4 | 38567.4 |

$$v'_i = C_i^{k'_i} + \beta \sum_j p_{ij}^{k'_i} v_j \qquad i=2,\ k'_i=2$$

$$\begin{aligned} v'_2 &= 2100 + 0.98522\,(0.75v_1 + 0.1875v_2 + 0.675v_3) \\ &= 2100 + 0.98522\,(0.75\times35567.4 + 0.1875\times36960.8 \\ &\qquad + 0.0625\times38299.4) \\ &= 37567.40 \end{aligned}$$

That is, if we are initially in state 2 and replace the machine, but thereafter follow the original policy R, the present value of all future costs is $27,567.40



# Policy Improvement

| k | name | V' | ΔV |
|---|---|---|---|
| 1 | Keep | 36960.8 | 0 |
| 2 | Replace | 37567.4 | 606.62 |

V'(k) = total discounted cost if action k is selected for one stage, & current policy is followed thereafter

ΔV(k) = improvement (if <0)

Since $v'_2 > v_2$, the current policy for this state should not be changed.

# Policy Improvement

State #3, Major deterioration

Current Policy: action #1, Keep

Evaluate the alternate action: Replace

| k | name | V' | ΔV |
|---|---|---|---|
| 1 | Keep | 38299.4 | 0 |
| 2 | Replace | 38067.4 | ⁻232.035 |

Since $v'_3 < v_3$, i.e., $\Delta v_3 = v'_3 - v_3 < 0$, the policy in this state should be changed to "replace"



# Value Determination

Discount factor = 0.985222
(rate of return = 1.5%)

*New policy:*

| State | | Action |
|---|---|---|
| 1 | Good as new | 1 Keep |
| 2 | Minor deterioration | 1 Keep |
| 3 | Major deterioration | 2 Replace |
| 4 | Failed | 2 Replace |

*New present values:*

| i | $V_i$ |
|---|---|
| 1 | 33799.4 |
| 2 | 35128.6 |
| 3 | 36299.4 |
| 4 | 36799.4 |

# Policy Improvement

State #2, Minor deterioration

Current Policy: action #1, Keep

| k | name | V' | ΔV |
|---|---|---|---|
| 1 | Keep | 35128.6 | 0 |
| 2 | Replace | 35799.4 | 670.718 |

V'(k) = total discounted cost if action k is selected for one stage, & current policy is followed thereafter

ΔV(k) = improvement (if <0)

*The policy for this state should not be changed.*



# Policy Improvement

State #3, Major deterioration

Current Policy: action #2, Replace

| k | name | V' | ΔV |
|---|---|---|---|
| 1 | Keep | 36386.1 | 86.709 |
| 2 | Replace | 36299.4 | 0 |

V'(k) = total discounted cost if action k is selected for one stage, & current policy is followed thereafter

ΔV(k) = improvement (if <0)

*The policy for this state should not be changed.*

# Policy Improvement

No improvement is possible, so the current policy:

| State | | Action | |
|---|---|---|---|
| 1 | Good as new | 1 | Keep |
| 2 | Minor deterioration | 1 | Keep |
| 3 | Major deterioration | 2 | Replace |
| 4 | Failed | 2 | Replace |

is optimal!