Pre-Calculus Facility Location Problems

author

- "The Stream-side Tavern" Location Problem
- Locating a Market for Three Villages
- Locating a Market for more than 3 Villages (Weber's Problem)
- The Ice-Cream Vendor Problem
- The Firehouse Problem
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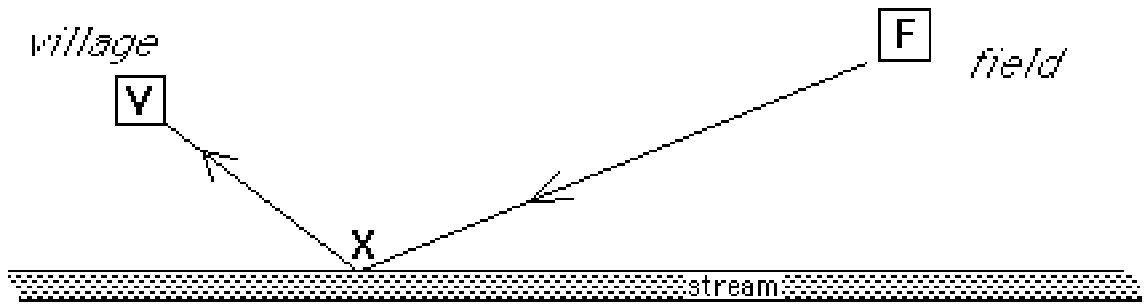
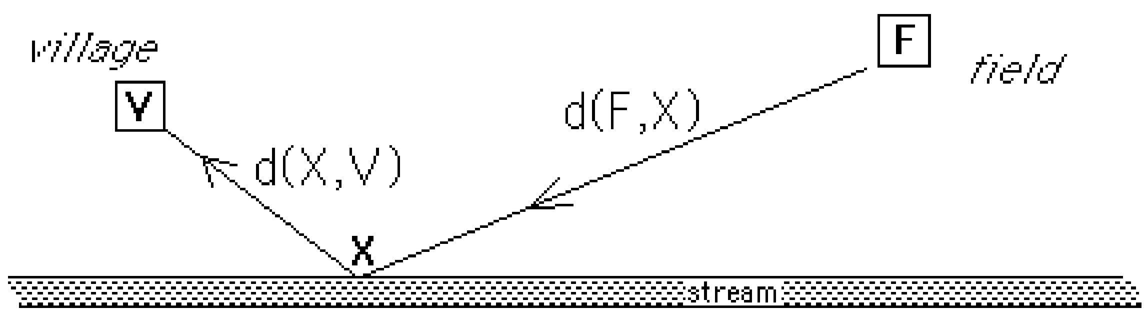
The Greek Streamside Tavern



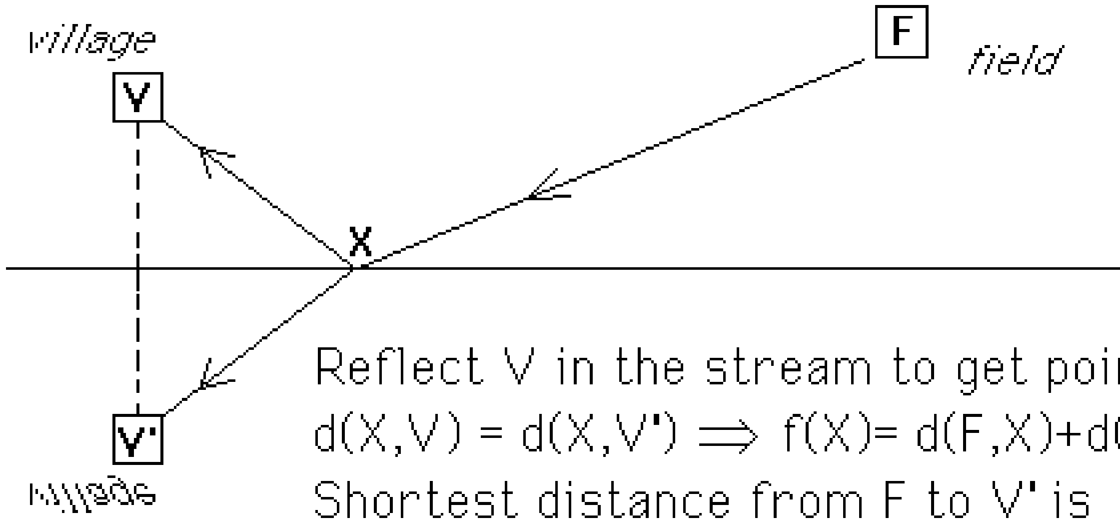
Villagers returning home after working in the communal field will walk to the stream to fill a bucket with water, and then walk to the village.

Where would be the best location for a tavern along the stream? (Assume X is the location which minimizes the total distance walked!)

Minimize $f(X) = d(F,X) + d(X,V)$



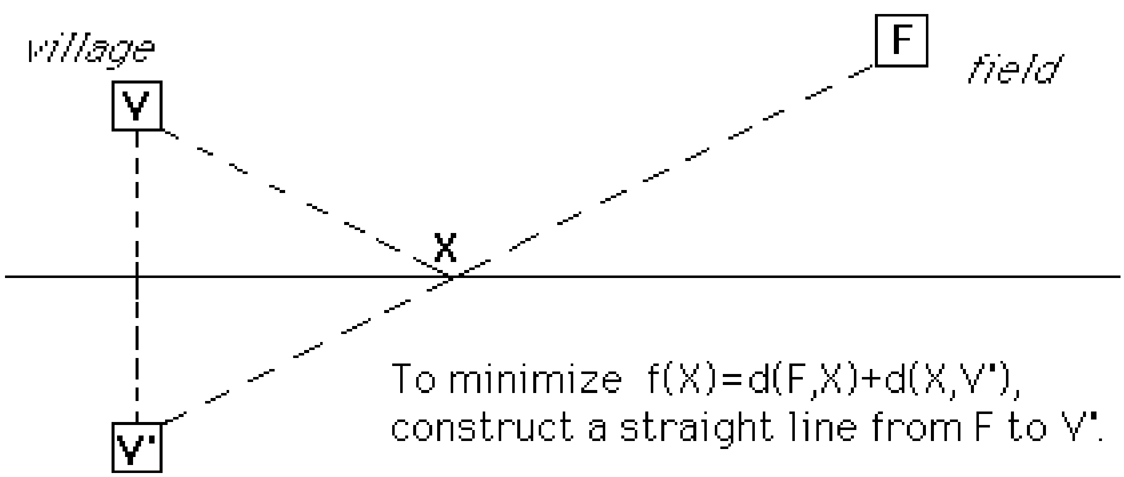
Geometric solution:



Reflect V in the stream to get point V'

$d(X,V) = d(X,V') \Longrightarrow f(X) = d(F,X) + d(X,V')$

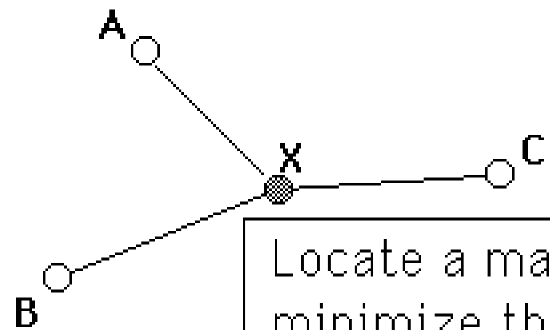
Shortest distance from F to V' is along a straight line!



To minimize $f(X)=d(F,X)+d(X,V')$, construct a straight line from F to V'.

The intersection with the stream is the location X which minimizes f(X).

Locating a marketplace for 3 villages

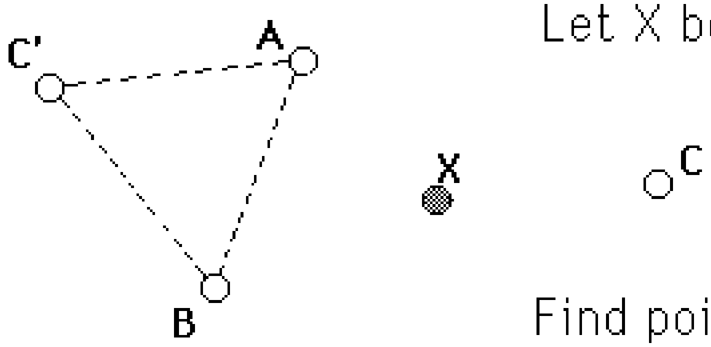


Locate a marketplace (X) so as to minimize the sum of the distances from X to each of three villages (A, B, & C).

i.e., select X to minimize d(X,A)+d(X,B)+d(X,C)

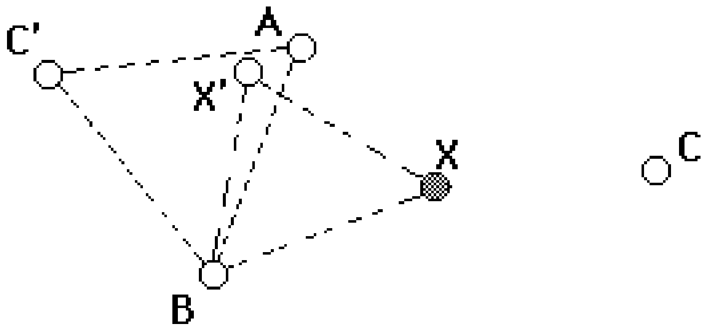
Geometric solution

Let X be an arbitrary point.

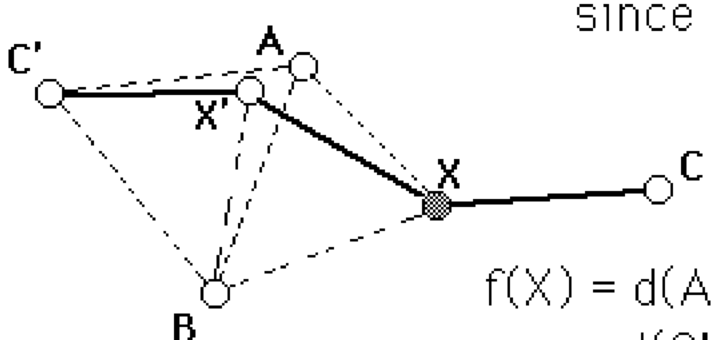


Find point C' so that
△ABC' is equilateral.

Find point X' so that
$\triangle BXX'$ is equilateral.



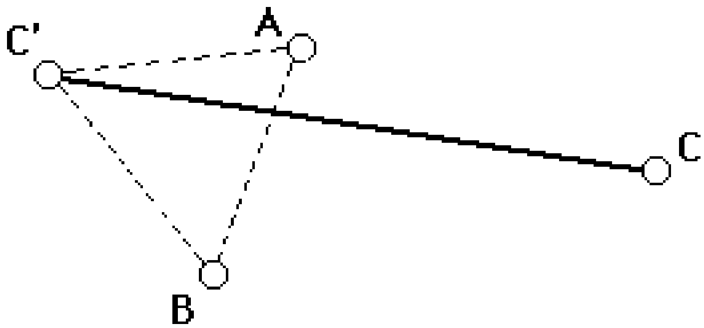
$\triangle BXA \cong \triangle BX'C'$
since $\angle C'BX' = \angle ABX$
$C'B = AB$
& $BX' = BX$



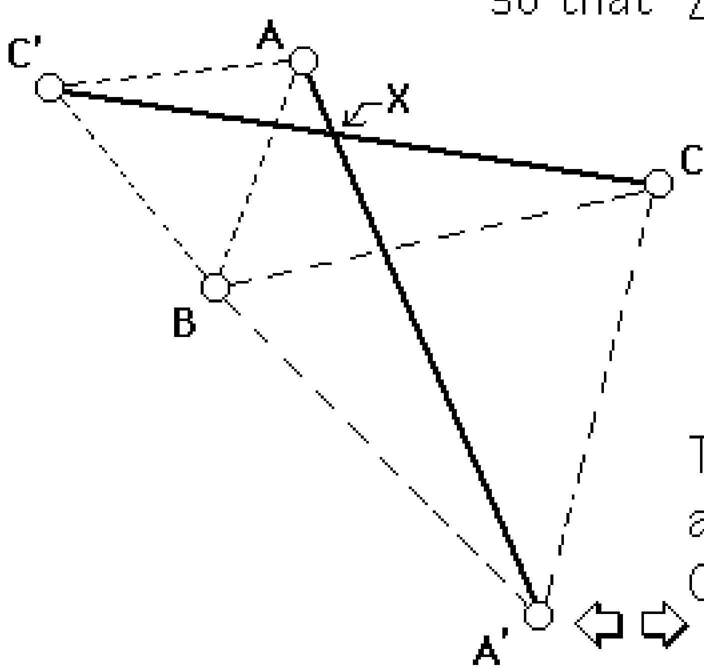
$$
\begin{aligned}
f(X) &= d(A,X) + d(B,X) + d(C,X) \\
&= d(C',X') + d(X',X) + d(X,C) \\
&= \text{length of the path } C'X'XC
\end{aligned}
$$

$\therefore$ f(X) is minimized if C'X'XC are colinear,
i.e., if X is on the line segment C'C

X is therefore found by constructing
C' so that △ABC' is equilateral.



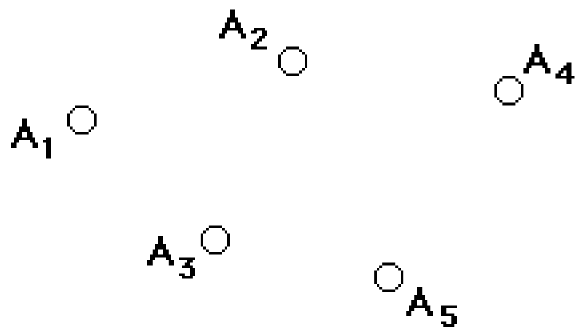
Next, point A' is constructed
so that △A'BC is equilateral.



By symmetry of
argument, X must be
on the line segment
A'A.

Therefore, X must lie
at the intersection of
C'C and A'A

Locating Marketplace for $n>3$ Villages:

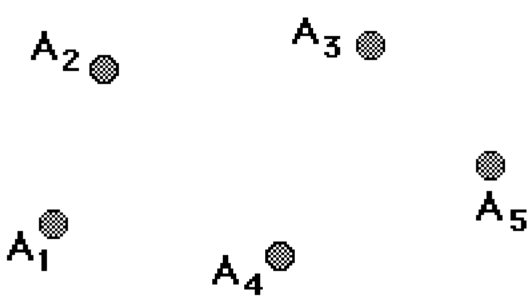


The geometric construction for the case n=3 does not generalize for n>3

("Weber's Problem")

Suppose that the "weight" at village i is W_i

We wish to minimize $f(X) = \sum_{i=1}^{n} W_i d(A_i, X)$



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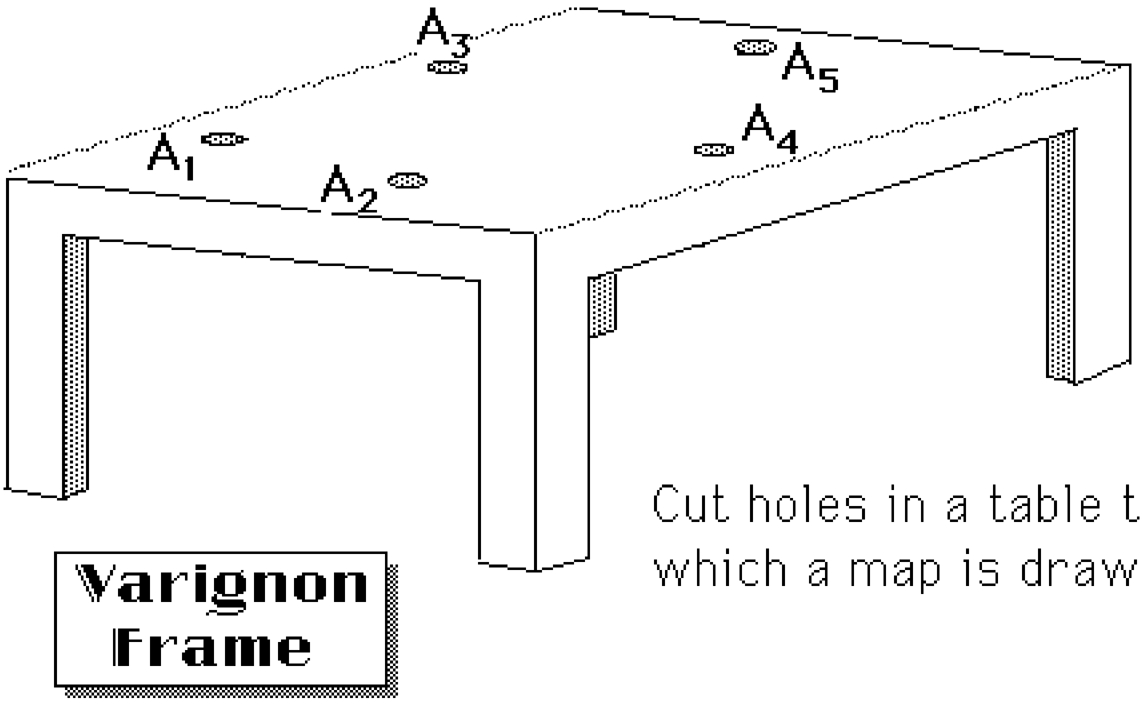
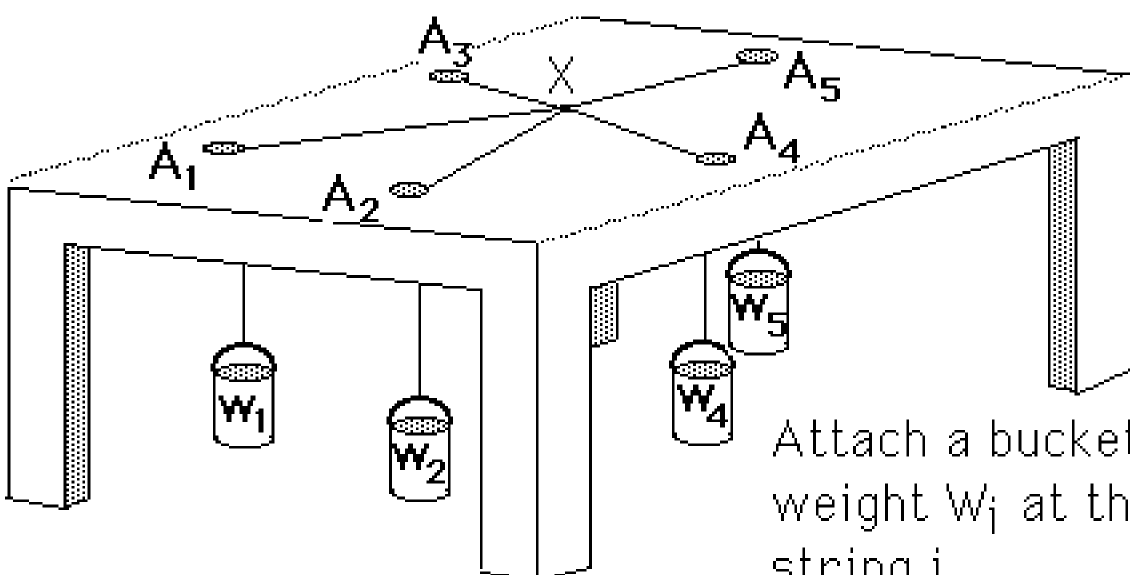
An "analogue computer":



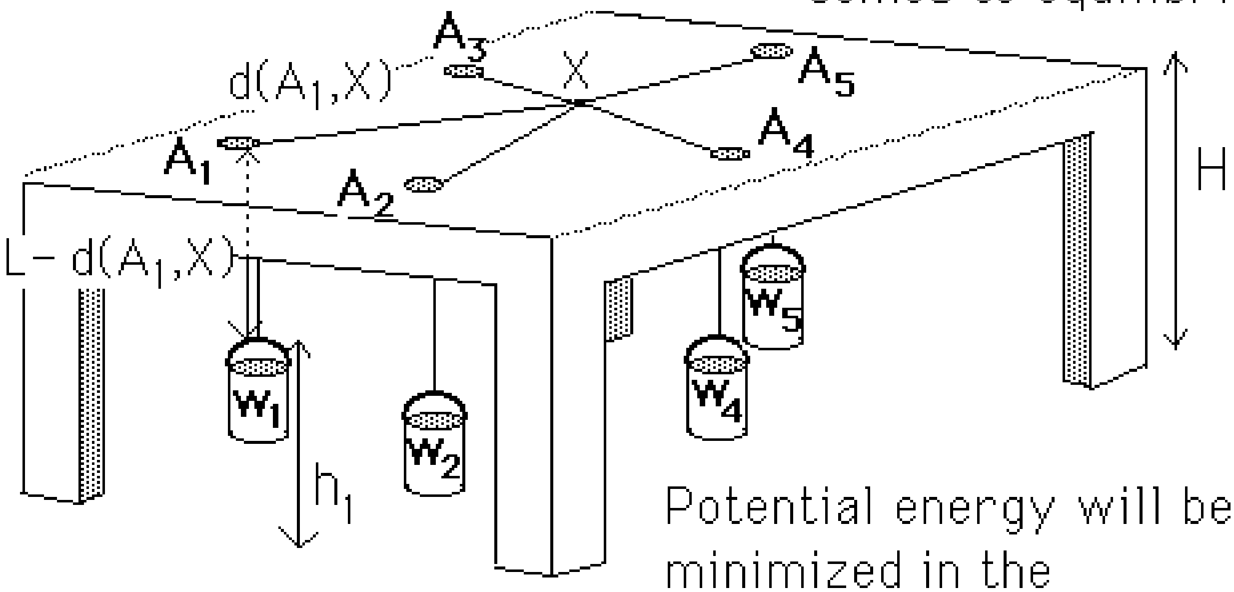
Cut holes in a table top on which a map is drawn.

Tie together in a knot ("X") n strings of equal length L



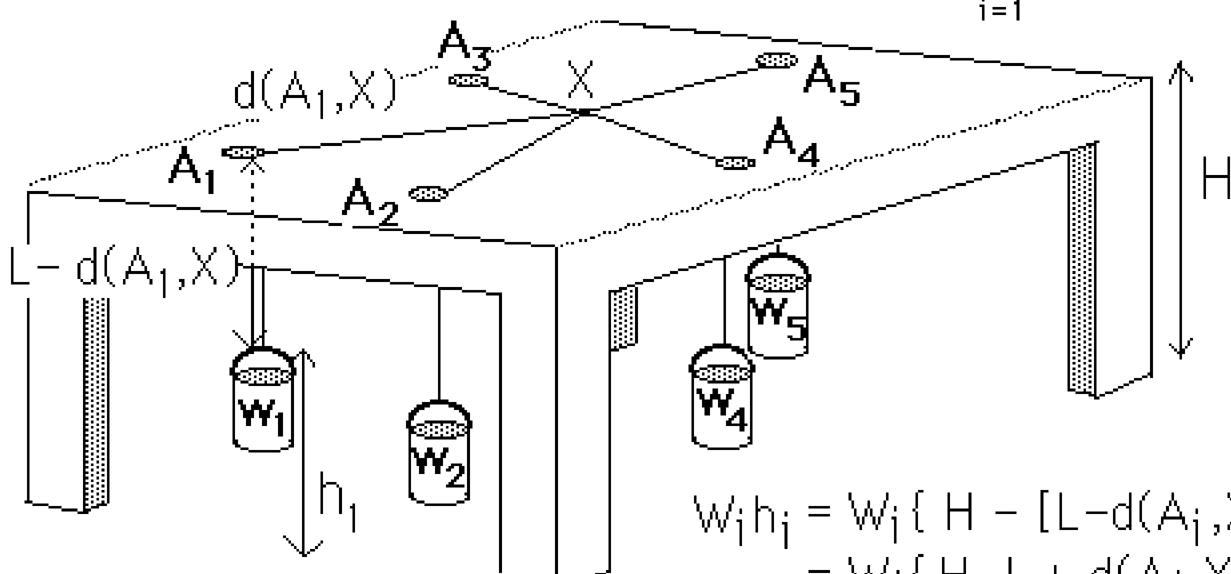
Attach a bucket with weight W_i at the end of string i.

The system of strings & weights is released and comes to equilibrium.



Potential energy will be minimized in the equilibrium state!

Potential Energy of System is $P.E. = \sum_{i=1}^{n} W_i h_i$



$$W_i h_i = W_i \{ H - [L - d(A_i,X)] \}$$
$$= W_i \{ H - L + d(A_i,X) \}$$
$$= (H-L)W_i + W_i d(A_i,X)$$

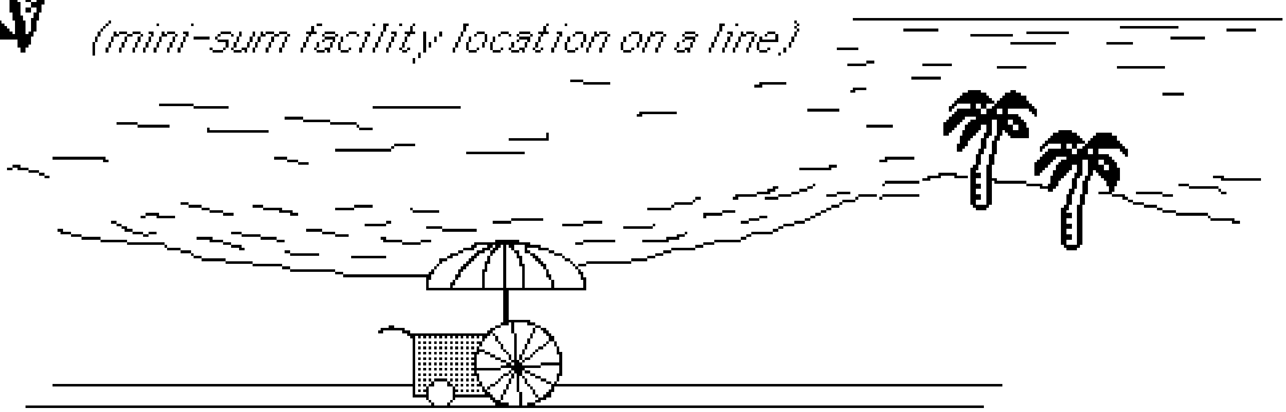
$$P.E. = \sum_{i=1}^{n} W_i h_i = \sum_{i=1}^{n} (H-L) W_i + W_i d(A_i, X)$$

$$= \underbrace{\sum_{i=1}^{n} (H-L) W_i}_{constant} + \underbrace{\sum_{i=1}^{n} W_i d(A_i, X)}_{f(X)}$$

The X which minimizes the potential energy will minimize our objective function!

The Ice-Cream Vendor Problem

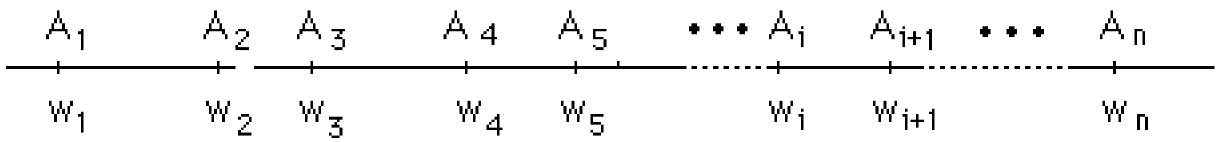
(mini-sum facility location on a line)



An ice cream vendor wants to locate his cart along a beach, minimizing the total distance from the persons on the beach to his cart.

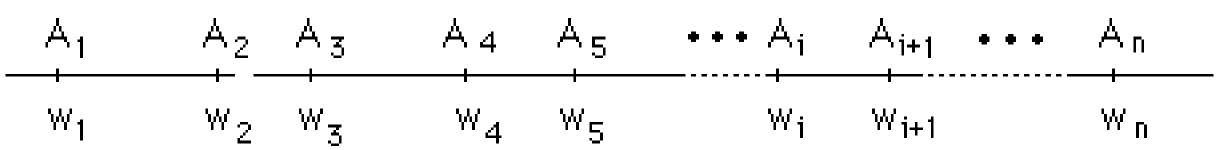
Consider the beach to be a line, with points representing groups of people: A_i = location of group #i

W_i = # of persons in group #i



His objective: Minimize $f(x) = \sum_i w_i \left| x - A_i \right|$

What is the nature of this objective function, f(x)?



If x lies in the interval $[A_i, A_{i+1}]$,

$$f(x) = \sum_{j \le i} w_j (x - A_j) + \sum_{j > i} w_j (A_j - x)$$

$$= \sum_{j \le i} w_j x - \sum_{j \le i} w_j A_j + \sum_{j > i} w_j A_j - \sum_{j > i} w_j x$$

$$= x \left(\sum_{j \le i} w_j - \sum_{j > i} w_j \right) + \underbrace{\left(\sum_{j > i} w_j A_j - \sum_{j \le i} w_j A_j \right)}_{\textit{a constant!}}$$

We have seen that on the interval

$$f(x) = x\left(\sum_{j \le i} w_j - \sum_{j > i} w_j\right) + \text{(constant)}$$

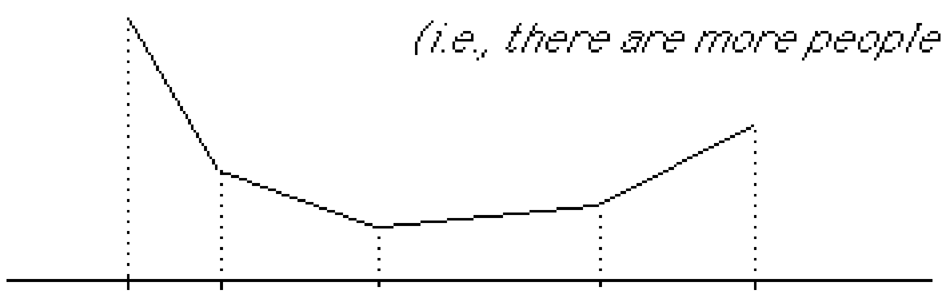
This is a linear function, with slope $\sum_{j \le i} w_j - \sum_{j > i} w_j$

Therefore, the function is increasing if $\sum_{j \le i} w_j > \sum_{j > i} w_j$

and decreasing if $\sum_{j \le i} w_j < \sum_{j > i} w_j$

That is, the function is increasing if $\sum_{j \le i} w_j > \sum_{j > i} w_j$

(i.e., there are more people to the left.)



and decreasing if $\sum_{j \le i} w_j < \sum_{j > i} w_j$

(i.e., there are more people to the right.)

So the ice cream vendor should walk along the beach until half of the bathers are behind him, and half in front of him.

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Note that this means that the optimal solution is at the ***median*** of the population (not the ***mean***, as one might have suspected!)

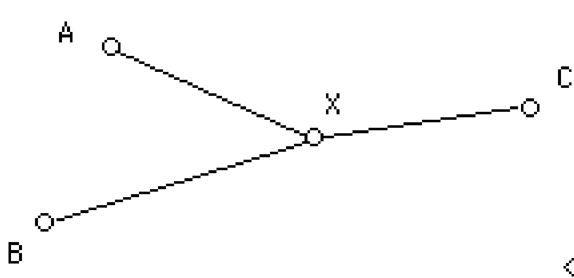
(For this reason, mini-sum location problems are referred to as ***median*** problems.)

Question: what happens to the optimal location if, during the afternoon, a group of bathers to the right of the cart decide to move 100 meters further to the right?

The Firehouse Location Problem

Suppose that a firehouse is to be located so as to serve three villages. The "best" location is considered to be that which would minimize the longest possible delay in reaching the scene of a fire:

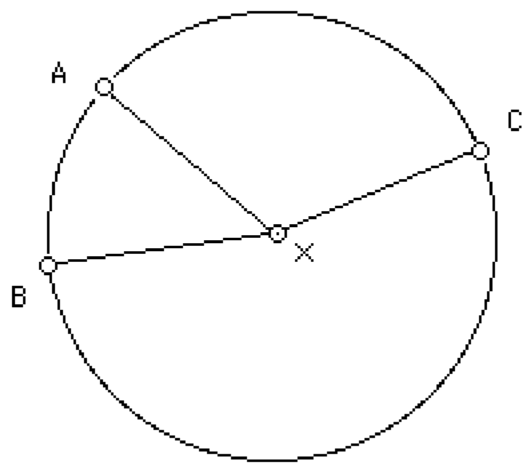
Minimize $g(x)$ where $g(x) = \text{maximum}\,\{d(x,A), d(x,B), d(x,C)\}$



$g(x) \le z \iff$ A, B, & C are contained within a circle of radius z with center x

The problem then is equivalent to finding the smallest circle which circumscribes the three points A, B, and C.

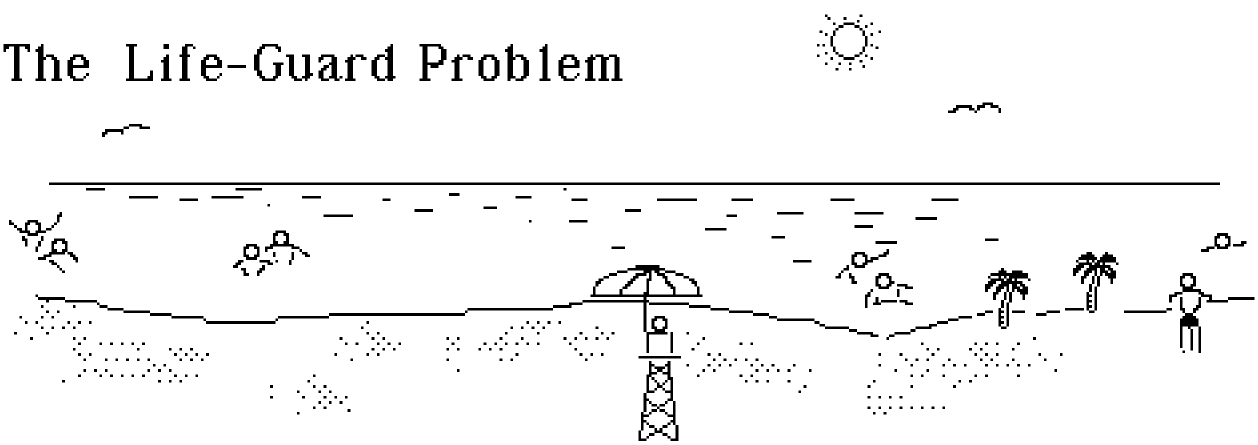
The optimal location is then at the ***center*** of the circle.



This easily generalizes to n villages, where n > 3.

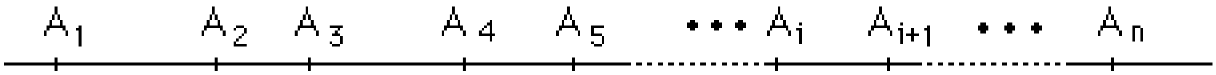
*For this reason, such "minimax" location problems are referred to as **center** problems!*

The Life-Guard Problem



The lifeguard wants to locate his chair in a location which will minimize, not the sum of distances to the swimmers, but the distance to the farthest swimmer.

Again we represent the beach by a line, with points indicating the position of the swimmers:



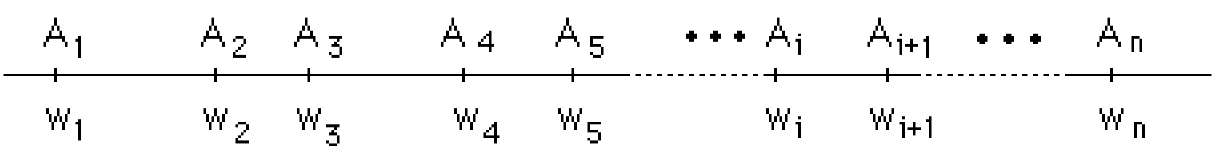
If his objective is to minimize f(x), where

$$f(x) = \underset{1 \le i \le n}{\text{maximum}} \; |x - A_i|$$

then the optimal location is the ***center*** of the smallest circle covering the n points.

☞ *Question: what happens to his optimal location if group #i decides to move to the right? What if i=n?*

Suppose that he wants to weight the distances, according to the number of persons in the group:



Let his objective function (to be minimized) be defined by

$$f(x) = \underset{1 \le i \le n}{\text{maximum}} \; w_i |x - A_i|$$

in APL: ⌈/W×|X−A

By definition of absolute value:

$$|x - A_i| = \text{maximum}\{(x-A_i), (A_i-x)\}$$

So our problem is

$$\underset{x}{\text{minimize}} \left[\underset{1\le i\le n}{\text{maximum}} \{(x-A_i), (A_i-x)\} \right]$$

$$\underset{x}{\text{minimize}} \left\{ \underset{1\le i\le n}{\text{maximum}} \quad w_i|x - A_i| \right\}$$

We can transform this optimization problem to an LP by introducing a new variable, z:

$$\underset{x,z}{\text{minimize}}\ z \qquad \text{subject to} \quad z \ge w_i|x - A_i| \qquad \text{for } i=1,\ldots n$$

$$\Rightarrow \underset{x,z}{\text{minimize}}\ z \qquad \text{subject to} \begin{cases} z \ge w_i(x - A_i) & \text{for } i=1,\ldots n \\ z \ge w_i(A_i - x) & \text{for } i=1,\ldots n \end{cases}$$

$$\Rightarrow \underset{x,z}{\text{minimize}}\ z \qquad \text{subject to} \begin{cases} x \le A_i + z/w_i & \text{for } i=1,\ldots n \\ x \ge A_i - z/w_i & \text{for } i=1,\ldots n \end{cases}$$

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The condition for a feasible solution to the constraints

$$\begin{cases} x \leq A_i + z/w_i & \text{for } i=1,\ldots n \\ x \geq A_i - z/w_i & \text{for } i=1,\ldots n \end{cases}$$

is, of course, that every upper bound of x be greater than (or equal to) every lower bound, that is,

$$A_i - z/w_i \leq A_j + z/w_j \quad \text{for every pair } i\&j$$

or, solving these inequalities for z,

$$A_i - A_j \leq z/w_j + z/w_i \Rightarrow \frac{A_i - A_j}{1/w_j + 1/w_i} \leq z \quad \text{for all } i\&j$$

Therefore, we wish to choose the smallest z for which the constraints $\frac{A_i - A_j}{1/w_j + 1/w_i} \leq z$ are feasible

Clearly, then, the optimal z will be the greatest of its lower bounds, i.e.,

$$z = \underset{i,j}{\text{maximum}} \left[\frac{A_i - A_j}{1/w_j + 1/w_i} \right]$$

What is then the optimal value for the location, x?

Suppose that the greatest lower bound for z occurs for the pair of points i^* and j^*. Then

$$z = \frac{A_i - A_j}{1/w_j + 1/w_i} \implies A_j - z/w_j \le x \le A_i + z/w_i$$

is the "tight" constraint,

that is, $A_j - z/w_j = x = A_i + z/w_i$

and, substituting the optimal value for z, we get

$$x = A_j - \frac{1}{w_j}\left[\frac{A_i - A_j}{1/w_j + 1/w_i}\right] = \frac{A_i w_i + A_j w_j}{w_i + w_j}$$

Notice that if the weights are equal, e.g., $w = 1$ for all j, then the optimal value is

$$z = \underset{i,j}{\text{maximum}}\left[\frac{A_i - A_j}{1/w_j + 1/w_i}\right] = \underset{i,j}{\text{maximum}}\left[\frac{A_i - A_j}{2}\right] = \frac{A_n - A_1}{2}$$

(i.e., $i^*=n$ and $j^*=1$)

and the optimal location is

$$x = \frac{A_i w_i + A_j w_j}{w_i + w_j} = \frac{A_n w_n + A_1 w_1}{w_n + w_1} = \frac{A_n + A_1}{2}$$

(i.e., at the midpoint of the interval $[A_1, A_n]$, the ***center***) which is what we would expect.

What is an APL expression for the optimal value in the weighted case, i.e.,

$$z = \underset{i,j}{\text{maximum}}\left[\frac{A_i - A_j}{1/w_j + 1/w_i}\right] \quad ?$$

⇦ ⇨

$$z = \underset{i,j}{\text{maximum}}\left[\frac{A_i - A_j}{1/w_j + 1/w_i}\right]$$

would be written in APL as:

```
z ← ⌈ / , ( A ∘.− A ) ÷ ( ÷ ⍵ ∘.+ ⍵ )
```

How could we write the expression for the optimal pair of indices, i and j*, used to maximize above?*

⇦ ⇨

The vector consisting of i and j* can be found by first locating the maximum value in the "raveled" matrix, i.e.,*

```
ij ← v⍳⌈/v ← ,( A ∘.- A )÷( ÷ w ∘.+ w )
```

and then using the "base" function to convert to row number, column number:

```
ij ← 1 + (2⍴⍴A) ⊥ ¯1 + ij
```

which could be combined into a single expression:

```
ij←1+(2⍴⍴A)⊥¯1+v⍳⌈/v←,(A∘.-A)÷(÷w∘.+w)
```


Then the optimal location, i.e.,

$$x = \frac{A_i w_i + A_j w_j}{w_i + w_j}$$

would be written in APL as:

```
x ← ( A[ij] +.× w[ij]) ÷ +/ w[ij]
```