Lagrangian Duality

author

Consider the inequality-constrained problem:

$$\begin{aligned}&\textbf{Minimize } f(x)\\&\textbf{subject to}\\&\qquad g_i(x) \le 0,\ i = 1, 2, \cdots m\\&\qquad\quad x \in X\end{aligned}$$

Define the Lagrangian function:

$$L(x,\lambda) = f(x) + \sum_{i=1}^{m} \lambda_i\, g_i(x)$$

Based upon this Lagrangian function, we define two functions:

$$L(x,\lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

Primal Objective

$$\bar{L}(x) \equiv \underset{\lambda \geq 0}{\text{Maximum}}\ L(x,\lambda)$$

Fix "x" and maximize with respect to the Lagrange multiplier

Dual Objective

$$\hat{L}(\lambda) \equiv \underset{x \in X}{\text{Minimum}}\ L(x,\lambda)$$

Fix the Lagrange multiplier and minimize w.r.t. "x"

$$L(x,\lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

$$\bar{L}(x) \equiv \underset{\lambda \geq 0}{\text{Maximum}}\ L(x,\lambda)$$

$$\hat{L}(\lambda) \equiv \underset{x \in X}{\text{Minimum}}\ L(x,\lambda)$$

Weak Duality Relationship: for all $x \in X$ and $\lambda \geq 0$,

$$\underset{\lambda \geq 0}{\text{Maximum}}\ L(x,\lambda) \equiv \bar{L}(x) \geq L(x,\lambda) \geq \hat{L}(\lambda) \equiv \underset{x \in X}{\text{Minimum}}\ L(x,\lambda)$$

primal objective *dual objective*

$$\boxed{L(x,\lambda) = f(x) + \sum_{i=1}^{m} \lambda_i\, g_i(x)}$$

Primal Objective

$$\bar{L}(x) \equiv \underset{\lambda \ge 0}{\text{Maximum}}\ L(x,\lambda)$$

$$= \begin{cases} f(x) & \text{if } g_i(x) \le 0\ \forall\ i \\ +\infty & \text{if } g_i(x) > 0 \text{ for some } i \end{cases}$$

If $g_i(x) \le 0\ \forall\ i$ *then optimal* λ_i*'s are zero;*

otherwise, if $g_i(x) > 0$ *for some i,* $L(x,\lambda)$ *is unbounded above as* $\lambda_i \rightarrow +\infty$

$$\boxed{L(x,\lambda) = f(x) + \sum_{i=1}^{m} \lambda_i\, g_i(x)}$$

Primal Problem

$$\boxed{\underset{x \in X}{\text{Minimize}}\ \bar{L}(x)}$$

where

$$\bar{L}(x) \equiv \underset{\lambda \ge 0}{\text{Maximum}}\ L(x,\lambda)$$

Dual Problem

$$\boxed{\underset{\lambda \ge 0}{\text{Maximize}}\ \hat{L}(\lambda)}$$

where

$$\hat{L}(\lambda) \equiv \underset{x \in X}{\text{Minimum}}\ L(x,\lambda)$$

Primal Problem $\boxed{\underset{x \in X}{\text{Minimize}}\ \bar{L}(x)}$

where $\bar{L}(x) = \begin{cases} f(x) & \text{if } g_i(x) \le 0\ \forall\ i \\ +\infty & \text{if } g_i(x) > 0 \text{ for some } i \end{cases}$

If there exists an x feasible in $\{g_i(x) \le 0\ \forall\ i\}$, then we can restrict our search for the minimizing x to such x's, and therefore

$$\underset{x \in X}{\text{Minimum}}\ \bar{L}(x) = \underset{x \in X}{\text{Minimum}}\ \{\ f(x)\ |\ g_i(x) \le 0\ \forall\ i\ \}$$

And so we see that

Primal Problem $\boxed{\underset{x \in X}{\text{Minimize}}\ \bar{L}(x)}$

is equivalent to our original problem:

$$\boxed{\begin{array}{l} \text{Minimize } f(x) \\ \text{subject to} \\ \qquad g_i(x) \le 0,\ i = 1, 2, \cdots m \\ \qquad\quad x \in X \end{array}}$$

Weak Duality Relationship

For all $x \in X$ and $\lambda \geq 0$,

$$\boxed{\bar{L}(x) \geq L(x,\lambda) \geq \hat{L}(\lambda)}$$

$$\left.\begin{matrix}\textit{primal}\\ \textit{objective}\end{matrix}\right\} \geq \left\{\begin{matrix}\textit{dual}\\ \textit{objective}\end{matrix}\right.$$

In particular, if x^* and λ^* are the primal and dual optima, respectively, then

$$\bar{L}(x^*) \geq \hat{L}(\lambda^*)$$

i.e., $\boxed{\bar{L}(x^*) - \hat{L}(\lambda^*) \geq 0}$ *Duality Gap*

Weak Duality Relationship

For all $x \in X$ and $\lambda \geq 0$,

$$\boxed{\bar{L}(x) \geq L(x,\lambda) \geq \hat{L}(\lambda)}$$

$$\left.\begin{matrix}\textit{primal}\\ \textit{objective}\end{matrix}\right\} \geq \left\{\begin{matrix}\textit{dual}\\ \textit{objective}\end{matrix}\right.$$

That is, any feasible dual solution gives a lower bound on all primal solutions, including of course the optimal.... this property is often used to advantage in branch-and-bound algorithms for combinatorial problems.

Definition $(\bar{x}, \bar{\lambda})$ is a *saddlepoint* of $L(x,\lambda)$

if $L(\bar{x},\bar{\lambda}) \le L(x,\bar{\lambda})\ \forall\, x \in X$

(which implies that $\bar{L}(\bar{x}) = L(\bar{x},\bar{\lambda})$)

and $L(\bar{x},\bar{\lambda}) \ge L(\bar{x},\lambda)\ \forall\, \lambda \ge 0$

(which implies that $\widehat{L}(\bar{\lambda}) = L(\bar{x},\bar{\lambda})$)

If $(\bar{x}, \bar{\lambda})$ is a saddlepoint of $L(x,\lambda)$

then $\boxed{\bar{L}(\bar{x}) = L(\bar{x}, \bar{\lambda}) = \widehat{L}(\bar{\lambda})}$

primal objective *dual objective*

so that the duality gap is zero!

EXAMPLE

$$\begin{aligned} &\text{Minimize } 4x_1^2 + 2x_1x_2 + x_2^2 \\ &\text{subject to } 3x_1 + x_2 \ge 6 \\ &\qquad x_1 \ge 0,\ x_2 \ge 0 \end{aligned}$$

Define:
$$g(x) = 6 - 3x_1 - x_2$$
$$X = \{ (x_1,x_2) \mid x_1 \ge 0,\ x_2 \ge 0 \}$$

The Lagrangian is
$$L(x,\lambda) = 4x_1^2 + 2x_1x_2 + x_2^2 + \lambda\,(6 - 3x_1 - x_2)$$

Dual objective:
$$\widehat{L}(\lambda) = \min_{x\ge 0} \{4x_1^2 + 2x_1x_2 + x_2^2 + \lambda\,(6 - 3x_1 - x_2)\}$$

The K-K-T necessary conditions for optimality of $x_1, x_2 \ge 0$ are:

(for λ fixed)

$$\frac{\partial L}{\partial x_1} = 8x_1 + 2x_2 - 3\lambda \ge 0$$
$$\frac{\partial L}{\partial x_2} = 2x_1 + 2x_2 - \lambda \ge 0$$
$$x_1\left[\frac{\partial L}{\partial x_1}\right] = 0, \quad x_2\left[\frac{\partial L}{\partial x_2}\right] = 0$$

with solution:
$$x_1^*(\lambda) = \lambda/3,\ x_2^*(\lambda) = \lambda/6$$
$$x_1, x_2 \ge 0\ \forall\ \lambda \ge 0$$

And so the dual objective is

$$\widehat{L}(\lambda) = L\left(\lambda/3,\ \lambda/6,\ \lambda\right)$$

$$= 6\lambda - \frac{7}{12}\lambda^2 \quad \longleftarrow \text{ a CONCAVE function of } \lambda$$

and the dual problem is

$$\boxed{\begin{array}{c}\textbf{Maximize } 6\lambda - \frac{7}{12}\lambda^2 \\ \textbf{subject to } \lambda \geq 0\end{array}}$$

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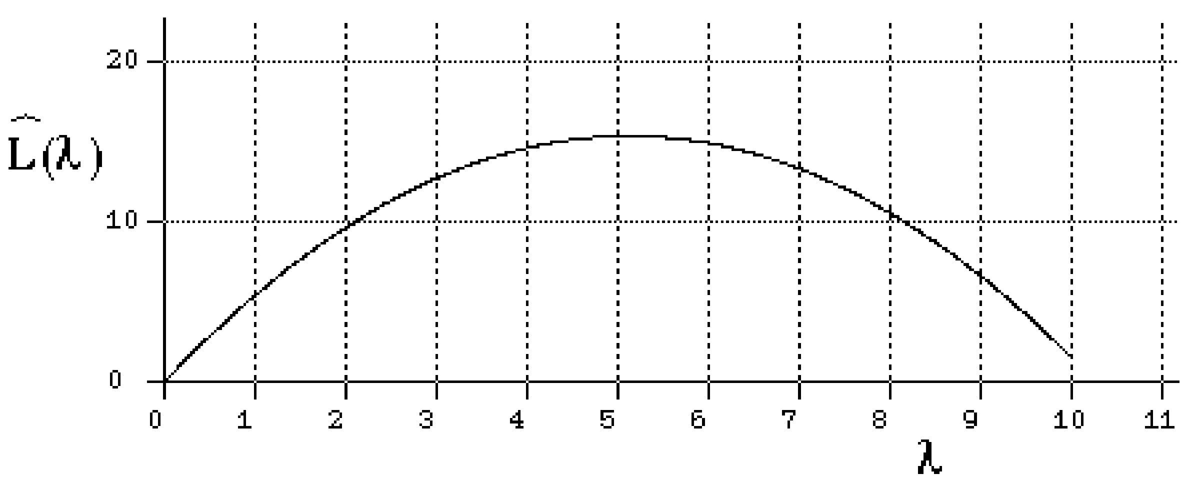


Dual problem:

$$\boxed{\begin{array}{c}\text{Maximize } 6\lambda - \frac{7}{12}\lambda^2 \\ \text{subject to } \lambda \ge 0\end{array}}$$

The necessary (& sufficient) conditions for optimality are

$$\frac{d\widehat{L}(\lambda)}{d\lambda} = 6 - 2\left(\frac{7}{12}\right)\lambda \le 0, \quad \lambda\left[\frac{d\widehat{L}(\lambda)}{d\lambda}\right] = 0$$

$$\Longrightarrow \quad \lambda^* = \frac{36}{7} \qquad \widehat{L}(\lambda^*) = \widehat{L}\left(\frac{36}{7}\right) = \frac{108}{7}$$

The corresponding values of x^* which optimize the Lagrangian subproblem, i.e., the problem of evaluating the dual objective $\widehat{L}$, are:

$$\begin{cases} x_1^*(\lambda^*) = \lambda^*/3 = \dfrac{36/7}{3} = \dfrac{12}{7}, \\ x_2^*(\lambda^*) = \lambda^*/6 = \dfrac{36/7}{6} = \dfrac{6}{7} \end{cases}$$

at which the primal objective, $4x_1^2 + 2x_1x_2 + x_2^2$, also has the value $\frac{108}{7}$

PRIMAL

$$\overline{L}(x) = \begin{cases} 4x_1^2 + 2x_1x_2 + x_2^2 & \text{if } 3x_1+x_2 \le 6,\ x \ge 0 \\ +\infty & \text{otherwise} \end{cases}$$

$$x_1^* = \frac{12}{7}, \quad x_2^* = \frac{6}{7}, \quad \overline{L}(x^*) = \frac{108}{7}$$

DUAL

$$\widehat{L}(\lambda) = 6\lambda - \frac{7}{12}\lambda^2, \qquad \lambda \ge 0$$

$$\lambda^* = \frac{36}{7}, \quad \widehat{L}(\lambda^*) = \frac{108}{7}$$

$$\overline{L}(x^*) = \widehat{L}(\lambda^*)$$

No Duality Gap!

Geometric Interpretation

primal

Minimize $f(x)$
subject to $g(x) \le 0$
$x \in X$

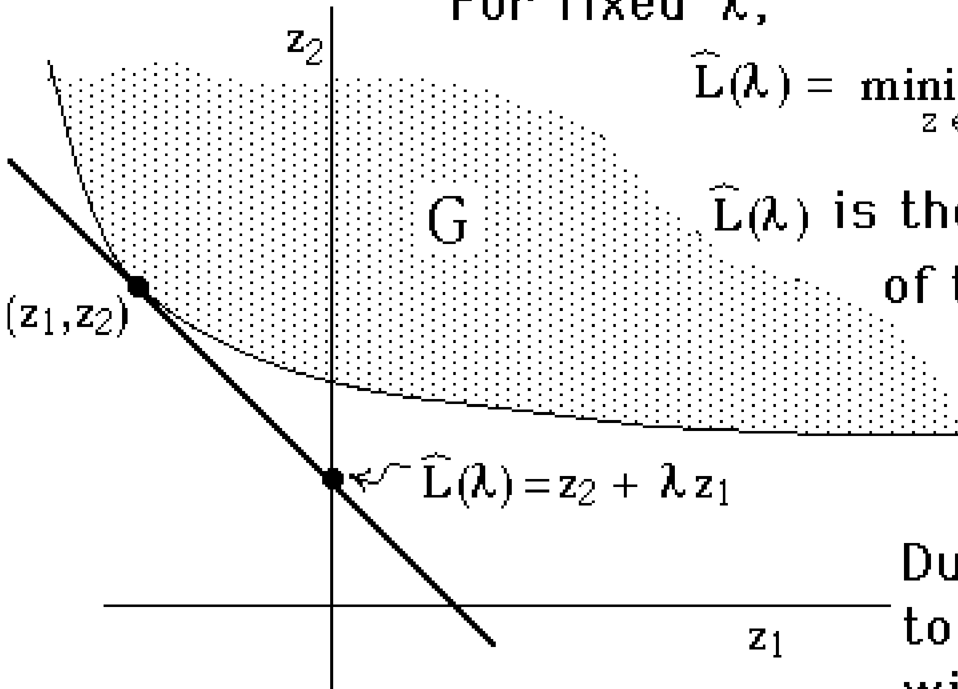
Define $G \equiv \{(z_1,z_2) \mid z_1 = g(x),\ z_2 = f(x) \text{ for } x \in X\}$

Primal can be restated as:

Minimize z_2
subject to
$z_1 \le 0$,
$z \in G$

For fixed λ,

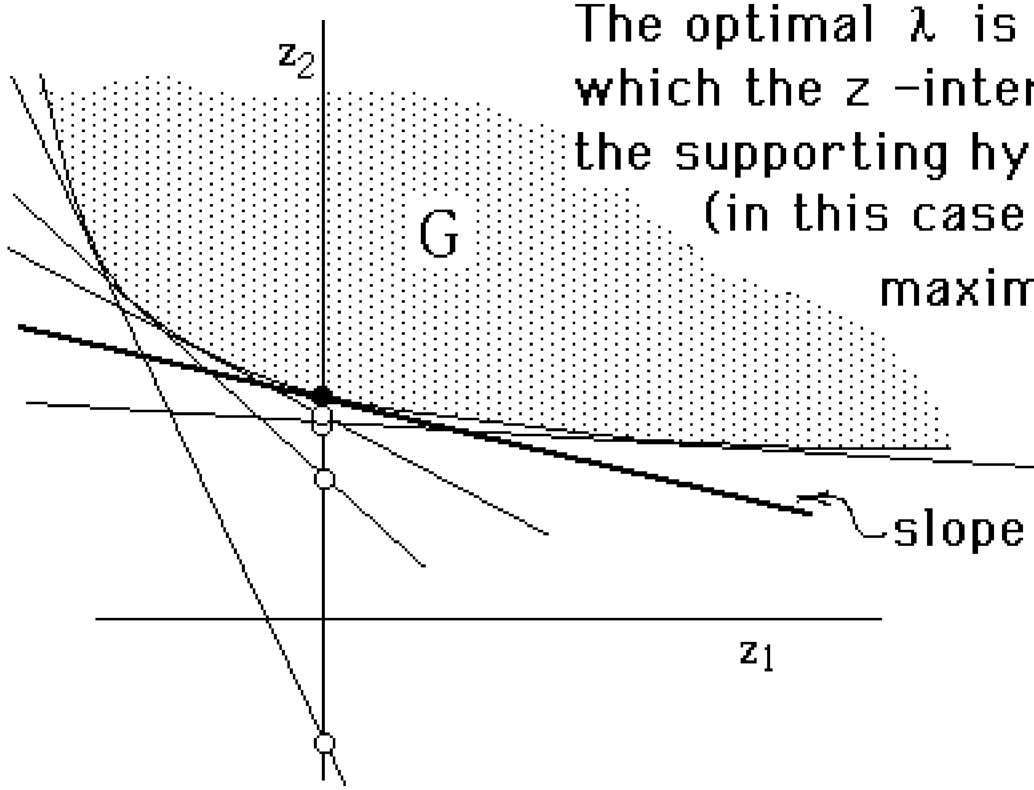
$$\hat{L}(\lambda) = \underset{z \in G}{\text{minimum}} \{ z_2 + \lambda z_1 \}$$



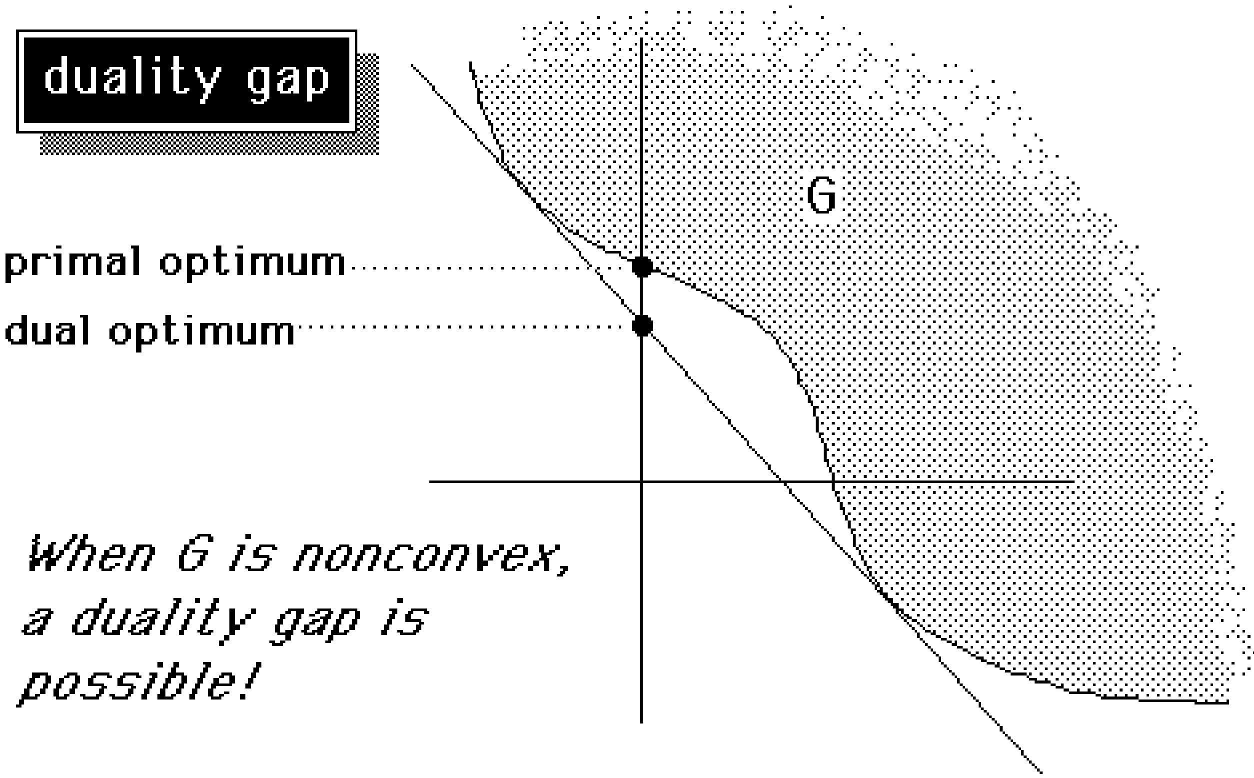
$\hat{L}(\lambda)$ is the z_2-intercept of the supporting hyperplane of G with slope $-\lambda$

Dual problem is to find support with maximum z_2-intercept

The optimal λ is that for which the z -intercept of the supporting hyperplane (in this case, line) is maximized.



duality gap



When G is nonconvex, a duality gap is possible!

EXAMPLE

integer linear program

$$\begin{cases} \text{Minimize } 3x_1 + 7x_2 + 10x_3 \\ \text{subject to } x_1 + 3x_2 + 5x_3 \geq 7 \\ x_j \in \{0,1\}\ ,\ j=1,2,3 \end{cases}$$

Define:

$$X \equiv \{\ x = (x_1, x_2, x_3\}\ |\ x_j \in \{0,1\}\ \}$$

$$= \{0,1\} \times \{0,1\} \times \{0,1\}$$ *Cartesian product*

$$g(x) \equiv 7 - x_1 - 3x_2 - 5x_3$$

Lagrangian function:

$$L(x,\lambda) = 3x_1 + 7x_2 + 10x_3 + \lambda(7 - x_1 - 3x_2 - 5x_3)$$

$$= (3 - \lambda)x_1 + (10 - \lambda)x_2 + (5 - \lambda)x_3 + 7\lambda$$

Dual objective:

$$\boxed{\hat{L}(\lambda) \equiv \underset{x_j \in \{0,1\}, j=1,2,3}{\text{Minimum}} \; L(x,\lambda)}$$

$$\hat{L}(\lambda) = \underset{x_j \in \{0,1\}}{\text{Minimum}} \; (3 - \lambda)x_1 + (10 - 3\lambda)x_2 + (5 - 5\lambda)x_3 + 7\lambda$$

Given a value of λ, the optimal $x_j^*(\lambda)$ is 0 if its coefficient is positive, and 1 otherwise.

For example, if $\lambda = 2.5$,

$$L(x,2.5) = 0.5x_1 - 0.5x_2 - 2.5x_3 + 17.5$$

$$x_1^*(2.5) = x_2^*(2.5) = 0, \;\; x_3^*(2.5)=1$$

$$\hat{L}(2.5) = 14.5$$

Thus,

$$x_1^*(\lambda) = \begin{cases} 1 & \text{if } 3 - \lambda \leq 0, \quad \text{i.e., } \lambda \geq 3 \\ 0 & \text{otherwise} \end{cases}$$

$$x_2^*(\lambda) = \begin{cases} 1 & \text{if } 7 - 3\lambda \leq 0, \quad \text{i.e., } \lambda \geq 7/3 \\ 0 & \text{otherwise} \end{cases}$$

$$x_3^*(\lambda) = \begin{cases} 1 & \text{if } 10 - 5\lambda \leq 0, \quad \text{i.e., } \lambda \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

will minimize $L(x,\lambda)$ for a given λ

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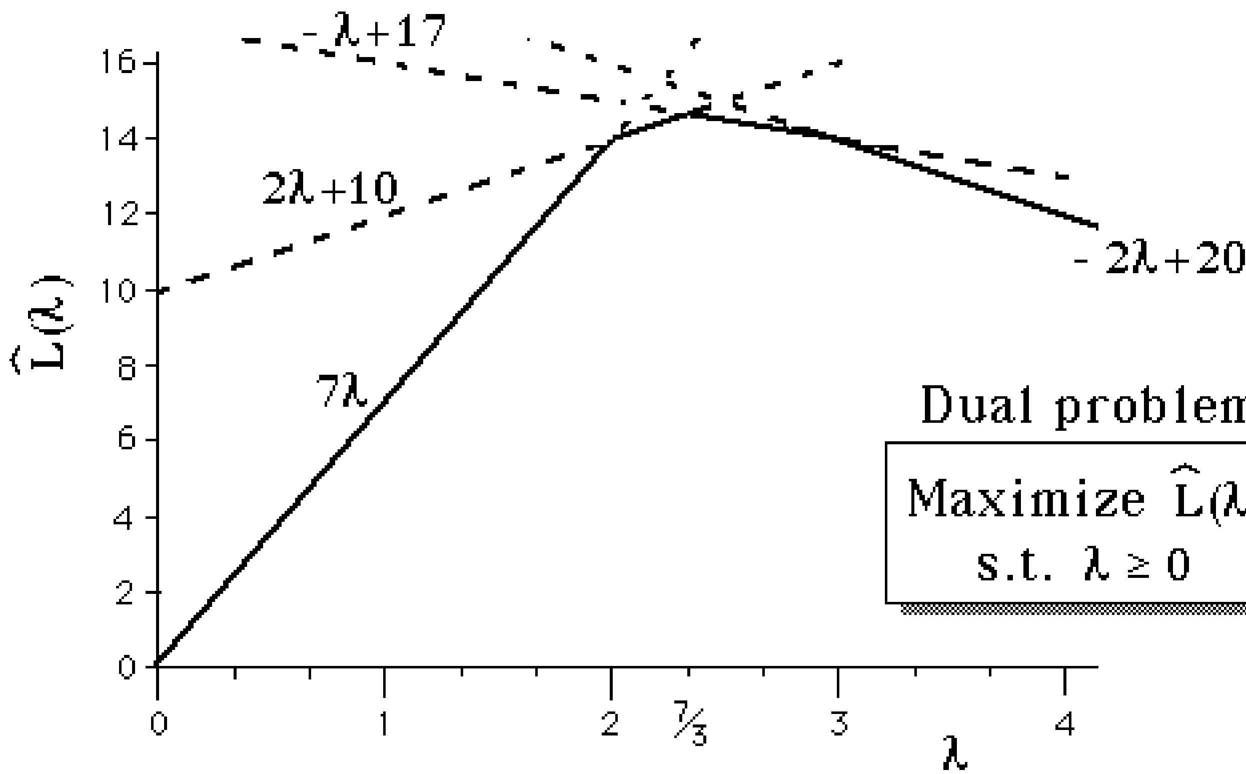
λ	$x_1^*(\lambda)$	$x_2^*(\lambda)$	$x_3^*(\lambda)$	$\hat{L}(\lambda)$
$0 \le \lambda \le 2$	0	0	0	7λ
$2 \le \lambda \le 7/3$	0	0	1	$2\lambda + 10$
$7/3 \le \lambda \le 3$	0	1	1	$-\lambda + 17$
$3 \le \lambda \le \infty$	1	1	1	$-2\lambda + 20$

When the coefficient of x_j is zero, then both 0 & 1 are optimal values for that variable.



Dual problem:

Maximize $\hat{L}(\lambda)$
s.t. $\lambda \ge 0$

By inspection of the graph of $\hat{L}(\lambda)$, we see that the optimal dual solution is

$$\lambda^* = 7/3 \quad , \qquad \hat{L}(\lambda^*) = 44/3$$

At λ^*, both $x' = (0,0,1)$ and $x'' = (0,1,1)$ minimize $L(x,\lambda)$.

But x' is infeasible in $x_1 + 3x_2 + 5x_3 \geq 7$

and x'' violates the complementary slackness condition:

$$\underbrace{\lambda^*}_{7/3} \underbrace{\left[7 - x_1'' - 3x_2'' - 5x_3''\right]}_{-1} \neq 0$$

Neither x' nor x'' are optimal in the primal problem!

Solving the primal problem by complete enumeration:

			Z_1	Z_2	
x_1	x_2	x_3	$g(x)$	$f(x)$	
0	0	0	7	0	infeasible
0	0	1	2	10	infeasible
0	1	0	4	7	infeasible
0	1	1	-1	17	☜ optimal in primal
1	0	0	6	3	infeasible
1	0	1	1	13	infeasible
1	1	0	3	10	infeasible
1	1	1	-2	20	feasible

Primal solution

$$\overline{L}(x'') = 17 = 51/3$$

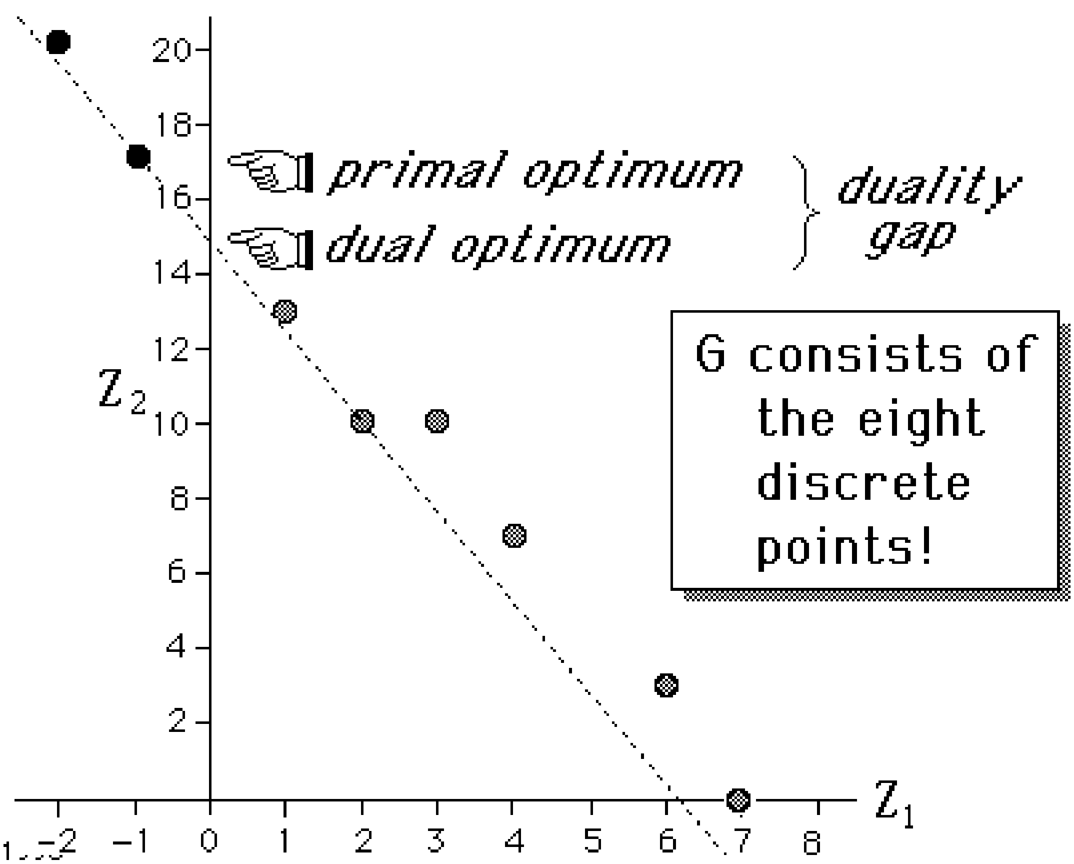
Dual solution

$$\widehat{L}(\lambda^*) = 44/3$$

Duality Gap > 0!

$$\overline{L}(x'') - \widehat{L}(\lambda^*) = 7/3$$

Graphical interpretation of the Duality Gap



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Saddlepoint Sufficiency Condition

Consider the problem:

> Minimize $f(x)$
> subject to
> $g_i(x) \leq 0, \ i = 1, 2, \cdots m$
> $x \in X$

where $f(x)$ & $g_i(x)$ are convex functions, and X is a convex set.

Let $\bar{\lambda} \geq 0$ and $\bar{x} \in X$....

Saddlepoint Sufficiency Condition

Then $(\bar{x}, \bar{\lambda})$ is a saddlepoint of the Lagrangian function $L(x, \lambda)$

if & only if

- $\bar{x}$ minimizes $L(x, \bar{\lambda}) = f(x) + \bar{\lambda}^T g(x)$ over X
- $g_i(\bar{x}) \leq 0$ for each $i = 1, 2, \cdots m$
- $\bar{\lambda}_i g_i(\bar{x}) = 0$ ← *which implies* $f(\bar{x}) = L(\bar{x}, \bar{\lambda})$

(If a saddlepoint exists, then the duality gap is zero!)

If $(\bar{x},\bar{\lambda})$ is a saddlepoint for $L(x,\lambda)$

then $\bar{x}$ solves the primal problem:

Minimize $f(x)$
subject to
$g_i(x) \leq 0, \; i = 1,2,\cdots m$
$x \in X$

and $\bar{\lambda}$ solves the dual problem:

Maximize $\hat{L}(\lambda)$
subject to $\lambda \geq 0$

where $\hat{L}(\lambda) \equiv \min_{x \in X} L(x, \lambda)$

STRONG DUALITY THEOREM

Consider the primal problem: Find

Φ = infimum $f(x)$
subject to $g_i(x) \leq 0, \; i = 1,2,\cdots m_1$
$h_i(x) = 0, \; i = 1,2,\cdots m_2$
$x \in X$

where $X \subseteq R^n$ is nonempty & convex
$f(x)$ & $g_i(x)$ are convex
$h_i(x)$ are linear

("infimum" may be replaced by "minimum" if the minimum is achieved at some x.)

STRONG DUALITY THEOREM

continued....

Define the Dual Problem:

Find

$$\Psi = \underset{\lambda \geq 0}{\text{supremum}}\ \widehat{L}(\lambda, \mu)$$

where

$$\widehat{L}(\lambda, \mu) \equiv \underset{x \in X}{\text{infimum}} \left\{ f(x) + \lambda^T g(x) + \mu^T h(x) \right\}$$

STRONG DUALITY THEOREM

continued....

Assume also that the following "Constraint Qualification" holds:

There exists $\hat{x}$ such that

$$g_i(\hat{x}) < 0,\ i = 1,2,\cdots m_1$$
$$h_i(\hat{x}) = 0,\ i = 1,2,\cdots m_2$$
$$\&\ 0 \in \text{int}\ h(X)$$

STRONG DUALITY THEOREM

continued....

Then $\boxed{\Phi = \Psi}$

i.e., there is no duality gap!

Furthermore, if $\Phi > -\infty$ then

- $\Psi = \hat{L}(\lambda^*, \mu^*)$ for some $\lambda^* \geq 0$
- if x^* solves the primal, it satisfies complementary slackness, i.e.,

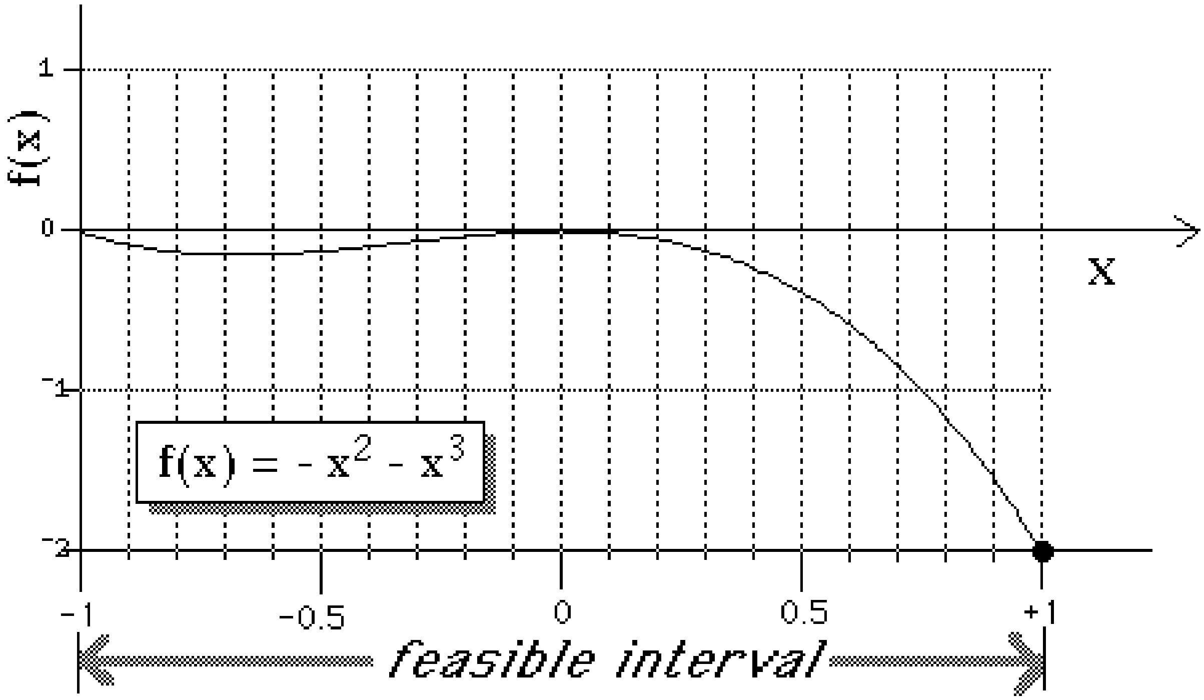
$$\lambda_i^* g_i(x^*) = 0 \ \forall \ i$$

EXAMPLE

$$\begin{array}{ll} \text{Minimize} & f(x) = -x^2 - x^3 \\ \text{subject to} & x^2 \leq 1 \end{array}$$

- Write the Lagrangian function
- State the KKT optimality conditions
- Solve graphically, and verify that the KKT conditions are satisfied at the optimum
- State the Lagrangian dual objective
- Solve the dual problem
- Is there a duality gap?

Graphically, we can see that $x^* = 1,\ f(x^*) = -2$



Lagrangian function

$$L(x,\lambda) = -x^2 - x^3 + \lambda\ (x^2 - 1)$$

KKT conditions

$$\frac{dL}{dx} = -2x - 3x^2 + 2\lambda\ x = 0$$

$$x^2 \leq 1$$

$$\lambda\ (x^2 - 1) = 0$$

$$\lambda \geq 0$$

KKT points are

$(x,\lambda) =$	$(-\frac{2}{3}, 0)$	$(0,0)$	$(1, \frac{5}{2})$
$L(x,\lambda) =$	$-\frac{4}{27}$	0	-2

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Dual Problem

Maximize $\widehat{L}(\lambda)$
subject to $\lambda \geq 0$

where $\widehat{L}(\lambda) \equiv \min\limits_{x \in X} L(x, \lambda)$

$$= \min_{x \in X} \left\{ -x^2 - x^3 + \lambda (x^2 - 1) \right\}$$

$$= -\infty \quad \text{for all } \lambda \geq 0$$

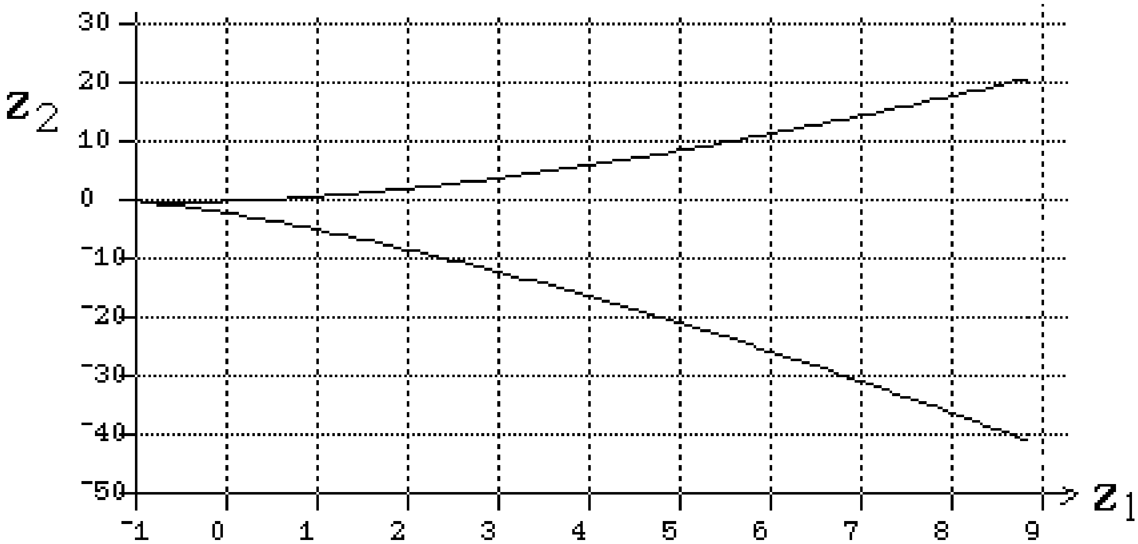
$$\Longrightarrow \quad \underset{\lambda \geq 0}{\text{Maximum}}\ \widehat{L}(\lambda) = -\infty$$

$$G = \left\{ (z_1, z_2) \mid z_1 = g(x),\ z_2 = f(x) \text{ for some } x \right\}$$

$$\begin{cases} z_2 = f(x) = -x^2 - x^3 \\ z_1 = g(x) = x^2 - 1 \quad \Rightarrow x = \pm (1 + z_1)^{1/2} \end{cases}$$

$$\Rightarrow \quad G = \left\{ (z_1, z_2) \mid z_2 = -(1 + z_1) \pm (1 + z_1)^{3/2} \right\}$$

The set G consists of the curve below:



There is no nonvertical support of G which has negative (= $-\lambda$) slope!

EXAMPLE

Minimize $-(x-4)^2$

subject to $1 \le x \le 6$

- Write the Lagrangian function
- State the KKT optimality conditions
- Solve graphically, and verify that the KKT conditions are satisfied at the optimum
- State the Lagrangian dual objective
- Solve the dual problem
- Is there a duality gap?

EXAMPLE

Minimize $f(x,y) = x$
subject to
$$g(x,y) = x^2 + y^2 \leq 1$$

- Write the Lagrangian function
- State the KKT optimality conditions
- Solve graphically, and verify that the KKT conditions are satisfied at the optimum
- State the Lagrangian dual objective
- Solve the dual problem
- Is there a duality gap?

EXAMPLE

Minimize $(x - 4)^2$
subject to
$$1 \leq x \leq 3$$

- Write the Lagrangian function
- State the KKT optimality conditions
- Solve graphically, and verify that the KKT conditions are satisfied at the optimum
- State the Lagrangian dual objective
- Solve the dual problem
- Is there a duality gap?