

## Example:

Because of excessive pollution on the Momiss River, the state is going to build some pollution control stations.

Three sites are under consideration.

The state is interested in controlling the pollution levels of two pollutants: #1 & #2

The legislature requires that at least 80,000 lbs. of pollutant #1 and at least 50,000 lbs. of pollutant #2 be removed from the river annually.



**Data:** *annual cost over lifetime of station*

Site #	Cost of building station	Cost of treating 1 ton H <sub>2</sub> O	Amt. removed per ton of water	
			Pollutant 1	Pollutant 2
1	\$ 100,000	\$20	0.4 lb.	0.3 lb.
2	\$ 60,000	\$30	0.25 lb.	0.2 lb.
3	\$ 40,000	\$40	0.2 lb.	0.25 lb.

Formulate an ILP to minimize the cost of meeting the state legislature's goals.

**Model**

## Decision Variables

continuous:

$X_i$  = tons of  $H_2O$  treated at station #i annually

binary:

$$Y_i = \begin{cases} 1 & \text{if a station is built at site #i} \\ 0 & \text{otherwise} \end{cases}$$

## Objective

$$\begin{aligned} \text{Minimize} \quad & 100000Y_1 + 60000Y_2 + 40000Y_3 \\ & \underbrace{\quad \quad \quad}_{\text{construction cost of stations}} \\ & + 20X_1 + 30X_2 + 40X_3 \\ & \underbrace{\quad \quad \quad}_{\text{water treatment costs}} \end{aligned}$$

## Constraints

The required amounts of pollutants are removed:

$$\begin{cases} 0.4X_1 + 0.25X_2 + 0.2X_3 \geq 80,000 & \text{pollutant 1} \\ 0.3X_1 + 0.2X_2 + 0.25X_3 \geq 50,000 & \text{pollutant 2} \end{cases}$$

Water cannot be treated at a station unless it has been built:

$$X_i \leq M Y_i \quad \forall i \quad \text{where } M \text{ is a suitably "big" number}$$

(If  $Y_i = 0$ , this forces  $X_i = 0$ )



## Example:

Glueco produces 3 types of glue on 2 different production lines.

Each line can be utilized by up to 7 workers at a time. Workers on production line #1 are paid \$500/week, and workers on line #2 are paid \$900/week.

It costs \$1000 to set up production line #1 for a week of production, and \$2000 for line #2.



**Data:**

	Weekly output per worker (Barrels)		
Line #	Glue #1	Glue #2	Glue #3
1	20	30	40
2	50	35	45
Weekly Rqmt:	120	150	200

Formulate an ILP to minimize the total cost of meeting weekly requirements.

**Model**

**Decision Variables**

*continuous:*

$X_{ij}$  = man-weeks of time on line  $i$  producing glue  $j$

*integer:*

$Y_i$  = number of persons assigned to line  $i$

*binary:*

$Z_i = \begin{cases} 1 & \text{if line } i \text{ is set up for production} \\ 0 & \text{otherwise} \end{cases}$

## Objective

$$\text{Minimize } \underbrace{1000 Z_1 + 2000 Z_2}_{\text{set-up costs}} + \underbrace{500 Y_1 + 900 Y_2}_{\text{labor costs}}$$

## Constraints

Requirements for glue are met:

$$20X_{11} + 50X_{21} \geq 120 \quad \text{glue } \#1$$

$$30X_{12} + 35X_{22} \geq 150 \quad \text{glue } \#2$$

$$40X_{13} + 45X_{23} \geq 200 \quad \text{glue } \#3$$

Capacity of line:

$$X_{11} + X_{12} + X_{13} \leq Y_1 \quad \text{line } \#1$$

$$X_{21} + X_{22} + X_{23} \leq Y_2 \quad \text{line } \#2$$

## Constraints

*We cannot assign workers to a line unless it has been set up:*

$$Y_i \leq 7 Z_i, i=1,2$$



### Example:

The manager of the University's computer system wants to be able to access 5 different files. There are multiple copies of each file, scattered among 10 disks, as shown below:

File #	Disk									
	1	2	3	4	5	6	7	8	9	10
1	X	X		X	X			X	X	
2		X		X						
3			X		X		X			X
4				X		X		X		
5	X	X		X		X	X		X	X



If disk 3 or disk 5 is used, then disk 2 must be used.

Formulate an ILP to select the smallest set of disks which contain at least once copy of each file.

**model**

## **Decision Variables**

$$x_i = \begin{cases} 1 & \text{if disk } \#i \text{ is selected} \\ 0 & \text{otherwise} \end{cases}$$

## **Objective**

$$\text{Minimize } X_1 + X_2 + \dots + X_{10}$$

## Constraints

*At least one copy of each file must be found among the selected disks:*

$$\begin{aligned}
 & X_1 + X_2 + X_4 + X_5 + X_8 + X_9 \geq 1 \\
 & X_1 + X_3 \geq 1 \\
 & X_2 + X_5 + X_7 + X_{10} \geq 1 \\
 & X_3 + X_6 + X_8 \geq 1 \\
 & X_1 + X_2 + X_4 + X_6 + X_7 + X_9 + X_{10} \geq 1
 \end{aligned}$$

## Constraints

*If disk 3 or disk 5 is used, then disk 2 must be used.*

$$\begin{cases} X_3 \leq X_2 \\ X_5 \leq X_2 \end{cases}$$

or

$$X_3 + X_5 \leq 2X_2$$

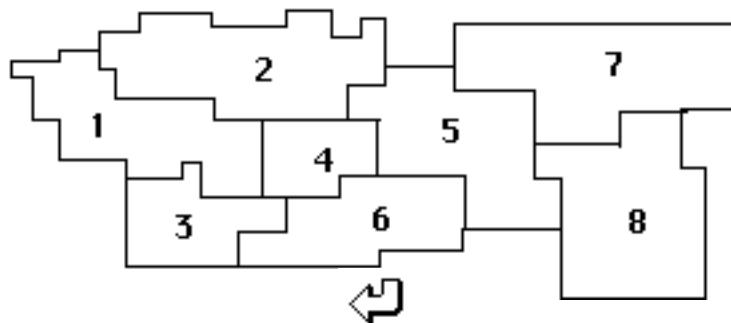
*(The pair of constraints is preferred from a computational point of view.)*



## Example

Gotham City has been divided into 8 districts.

The city wishes to station 2 ambulances so as to maximize the number of people who live within 3 minutes of an ambulance.



		to district								Population (thousands)
		1	2	3	4	5	6	7	8	
from district	1	0	3	4	6	8	9	8	10	40
	2	3	0	5	4	8	6	12	9	30
	3	4	8	0	2	2	3	5	7	35
	4	6	4	2	0	3	2	5	4	20
	5	8	8	2	3	0	2	2	4	15
	6	9	6	3	2	2	0	3	2	50
	7	8	12	5	5	2	3	0	2	45
	8	10	9	7	4	4	2	2	0	60

**Model**

## Decision Variables

The primary decision is whether to locate an ambulance in district  $i$  for  $i=1,2,\dots,8$ :

$$Y_i = \begin{cases} 1 & \text{if an ambulance is located in district } i \\ 0 & \text{otherwise} \end{cases}$$

Another set of decisions is whether to assign for service a district  $j$  to an ambulance location  $i$

$$X_{ij} = \begin{cases} 1 & \text{if district } j \text{ is to be served from district } i \\ 0 & \text{otherwise} \end{cases}$$

## Objective

We wish to maximize the number of persons within 3 minutes' travel time of an ambulance

For each district  $j$ , we need to include in the objective an expression of the form

$$P_j \sum \{X_{ij} : t_{ij} \leq 3\}$$

e.g., for  $j=3$ :

$$35(X_{33} + X_{43} + X_{53} + X_{63})$$

since districts 3,4,5,&6 are within 3 minutes' travel time of district 3.

## Objective

$$\begin{aligned} \text{Maximize } & 40(X_{11}+X_{21}) + 30(X_{12}+X_{22}) \\ & + 35(X_{33}+X_{43}+X_{53}+X_{63}) + 20(X_{34}+X_{44}+X_{54}+X_{64}) \\ & + 15(X_{35}+X_{45}+X_{55}+X_{65}+X_{75}) \\ & + 50(X_{36}+X_{46}+X_{56}+X_{66}+X_{76}+X_{86}) \\ & + 45(X_{57}+X_{67}+X_{77}+X_{87}) + 60(X_{68}+X_{78}+X_{88}) \end{aligned}$$

## Constraints

Every district must be assigned to an ambulance:

$$\sum_{i=1}^8 X_{ij} = 1 \quad \forall j=1,2,\dots,8$$

e.g.,

$$X_{11}+X_{21}+X_{31}+X_{41}+X_{51}+X_{61}+X_{71}+X_{81} = 1 \quad (j=1)$$

## Constraints

District  $j$  can be served from district  $i$  only if there is an ambulance in district  $i$ :

$$\sum_{j=1}^8 X_{ij} \leq 8Y_i \quad \text{for } i=1,2,\dots,8$$

e.g.,

$$X_{11} + X_{12} + X_{13} + X_{14} + X_{15} + X_{16} + X_{17} + X_{18} \leq 8Y_1$$

## Constraints

Two ambulances are to be assigned to districts:

$$\sum_{i=1}^8 Y_i = 2,$$

i.e.,

$$Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 + Y_7 + Y_8 = 2$$

## Model Statistics

# of binary variables:

Y: 8

X: 64

total: 72

# of continuous variables: 0

# of constraints: 17

## Another Formulation

Define as before:

$$Y_i = \begin{cases} 1 & \text{if an ambulance is located in district } i \\ 0 & \text{otherwise} \end{cases}$$

and

$$X_j = \begin{cases} 1 & \text{if district } j \text{ is within 3 minutes of} \\ & \text{an ambulance} \\ 0 & \text{otherwise} \end{cases}$$

## Objective

Maximize  $40X_1 + 30X_2 + 35X_3 + 20X_4 + 15X_5 + 50X_6$   
 $+ 45X_7 + 60X_8$

## Constraints

$X_j$  must be zero unless at least one ambulance is located within 3 minutes of district  $j$

$$X_1 \leq Y_1 + Y_2$$

$$X_2 \leq Y_1 + Y_2$$

$$X_3 \leq Y_3 + Y_4 + Y_5 + Y_6$$

•  
•  
•

$$X_8 \leq Y_6 + Y_7 + Y_8$$

## Constraints

Two ambulances are to be assigned to districts:

$$\sum_{i=1}^8 Y_i = 2,$$

i.e.,

$$Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 + Y_7 + Y_8 = 2$$

## Model Statistics

# of binary variables:

Y: 8

X: 8

total: 16

# of continuous variables: 0

# of constraints: 9

```
:MAX  40 X1 + 30 X2 + 35 X3 + 20 X4 + 15 X5 + 50 X6 +
45 X7 + 60 X8
SUBJECT TO
 2) X1 - Y1 - Y2 <= 0
 3) X2 - Y1 - Y2 <= 0
 4) X3 - Y3 - Y4 - Y5 - Y6 <= 0
 5) X4 - Y3 - Y4 - Y5 - Y6 - Y7 <= 0
 6) X5 - Y3 - Y4 - Y5 - Y6 - Y7 <= 0
 7) X6 - Y3 - Y4 - Y5 - Y6 - Y7 - Y8 <= 0
 8) X7 - Y5 - Y6 - Y7 - Y8 <= 0
 9) X8 - Y6 - Y7 - Y8 <= 0
10) Y1 + Y2 + Y3 + Y4 + Y5 + Y6 + Y7 + Y8 = 2
```

END

INTE 8

INTE Y1  
INTE Y2  
INTE Y3  
INTE Y4  
INTE Y5  
INTE Y6  
INTE Y7  
INTE Y8

: GO

LP OPTIMUM FOUND AT STEP 14

OBJECTIVE FUNCTION VALUE

1) 295.000000

VARIABLE	VALUE	REDUCED COST
X1	1.000000	-40.000000
X2	1.000000	-30.000000
X3	1.000000	-35.000000
X4	1.000000	-20.000000
X5	1.000000	-15.000000
X6	1.000000	-50.000000
X7	1.000000	-45.000000
X8	1.000000	-60.000000

VARIABLE	VALUE	REDUCED COST
Y1	1.000000	0.000000
Y2	0.000000	0.000000
Y3	0.000000	0.000000
Y4	0.000000	0.000000
Y5	0.000000	0.000000
Y6	1.000000	0.000000
Y7	0.000000	0.000000
Y8	0.000000	0.000000

NO. ITERATIONS= 14

BRANCHES= 0 DETERM.= 1.000E 0

FIX ALL VARS.( 8) WITH RC > 15.0000

LP OPTIMUM IS IP OPTIMUM

