Graphs and Networks: basic definitions & concepts

author

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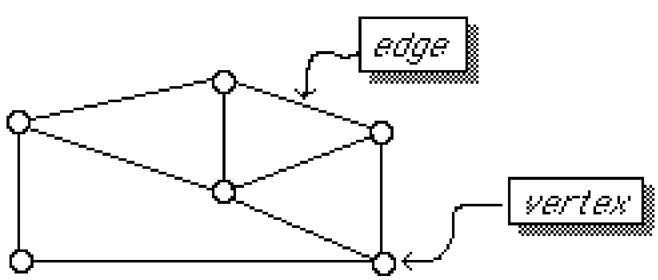
A GRAPH consists of

- a collection of VERTICES or NODES
- a collection of LINKS or EDGES

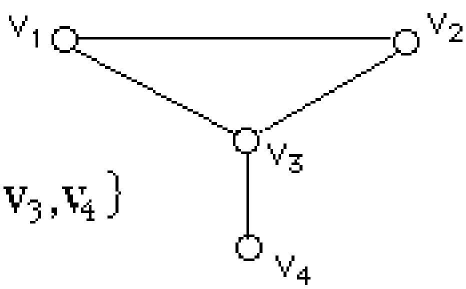


Formally, a GRAPH is a pair of sets (V,A) where

- V is non-empty
- A is an irreflexive, symmetric relation on V

irreflexive: $(v_i,v_i) \notin A$

symmetric: $(v_i,v_j) \in A \Leftrightarrow (v_j,v_i) \in A$



vertex set:

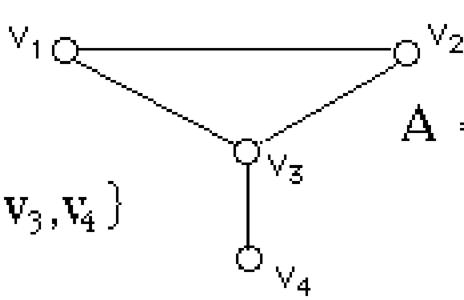
$V = \{v_1,v_2,v_3,v_4\}$

$$A = \{(v_1,v_2), (v_2,v_1), (v_2,v_3), (v_3,v_2), (v_3,v_4), (v_4,v_3), (v_1,v_3), (v_3,v_1)\}$$

a symmetric pair of elements $(v_i,v_j), (v_j,v_i)$ *is called an EDGE*

The number of vertices is the **ORDER** of the graph

The number of edges is the **SIZE** of the graph



vertex set:

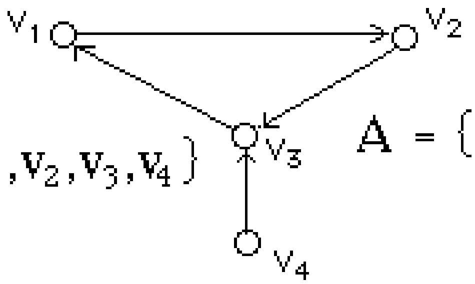
$V = \{v_1,v_2,v_3,v_4\}$

$$A = \{(v_1,v_2), (v_2,v_1), (v_2,v_3), (v_3,v_2), (v_3,v_4), (v_4,v_3), (v_1,v_3), (v_3,v_1)\}$$

ORDER=4

SIZE = 4

A **DIGRAPH** or DIRECTED GRAPH is a pair of sets (V,A) where A is not symmetric, that is, the links have directions

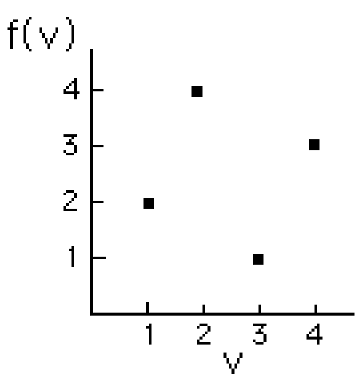


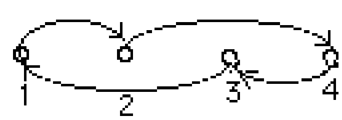
$V = \{v_1, v_2, v_3, v_4\}$

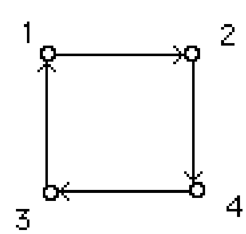
$A = \{(v_1,v_2), (v_2,v_3), (v_4,v_3), (v_3,v_1)\}$

Directed links are often called ARCS

Three representations of a digraph G=(V,A)
where V=(1,2,3,4) and A={(1,2), (2,4), (4,3), (3,1)}







f(1)=2,
f(2)=4,
f(3)=1,
f(4)=3

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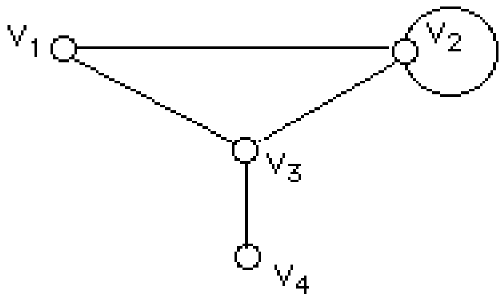
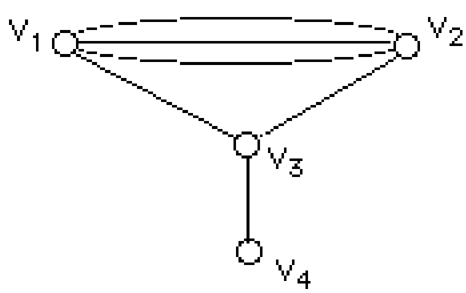
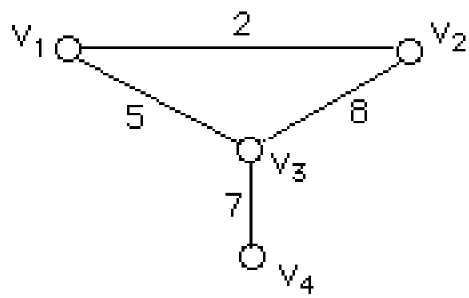
A "pure" graph has no loops, i.e., (v_i, v_i) is not a valid edge. If the edge set includes (v_i, v_i), the entity is called a LOOP-GRAPH



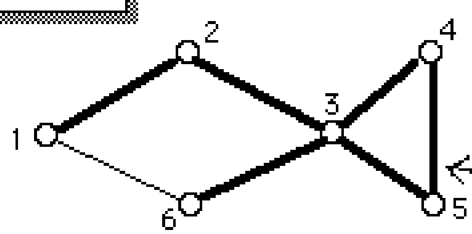
If multiple edges are allowed joining pairs of vertices, then the entity is called a MULTI-GRAPH



If each edge of a graph has an associated number, the entity is called a **NETWORK**

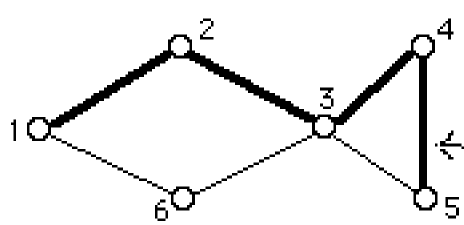


GRAPH



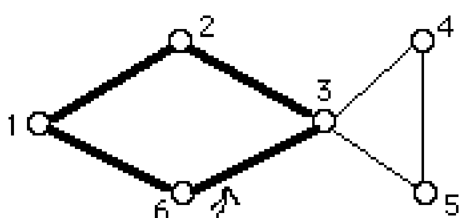
CHAIN : a sequence of vertices, $(x_0, x_1, \ldots x_i, x_{i+1}, \ldots x_s)$ where each pair (x_i, x_{i+1}) is an edge

(1,2,3,4,5,3,6)



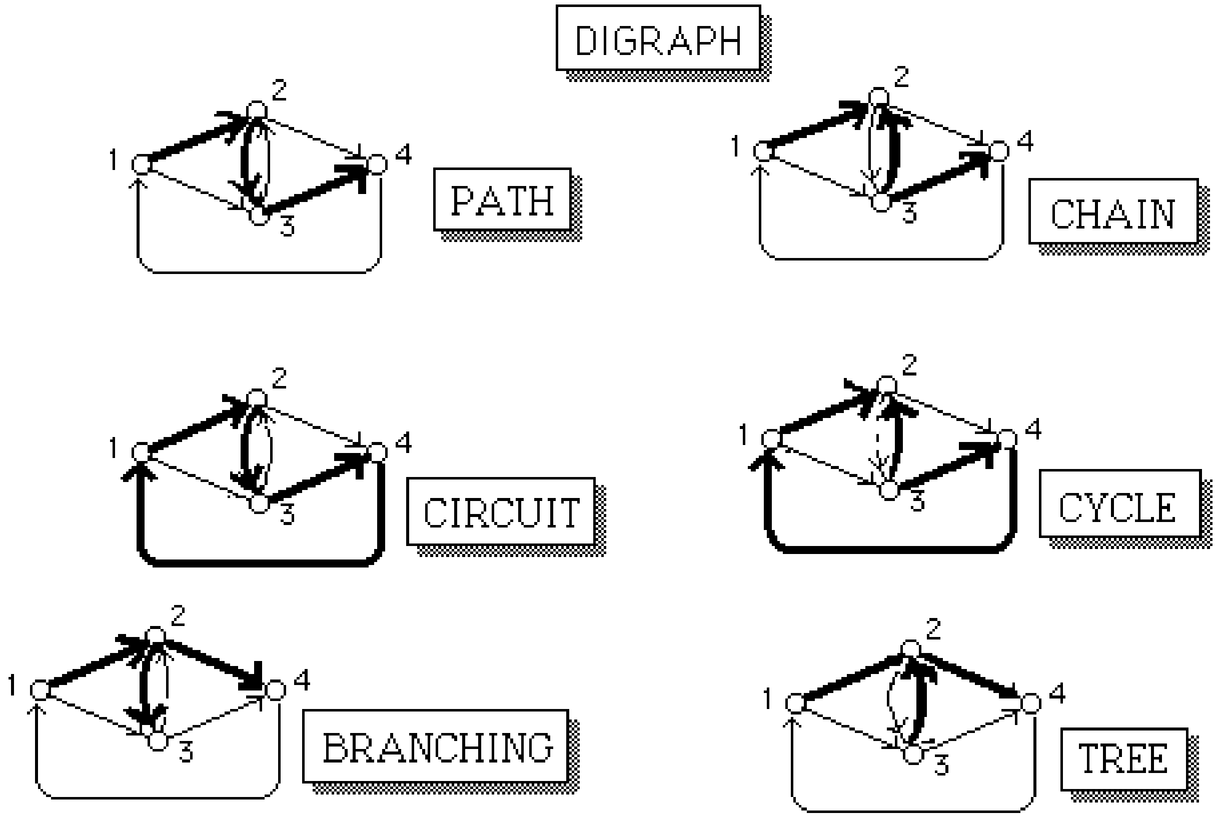
ELEMENTARY CHAIN (no vertices are repeated)

(1,2,3,4,5)

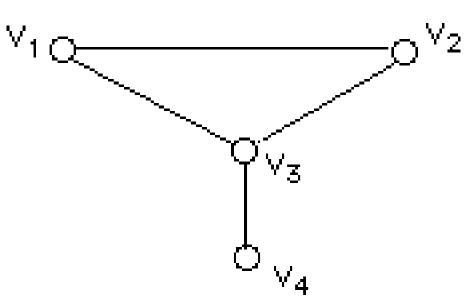


CYCLE (a closed chain, i.e., the first and last vertices of the chain are the same)

(1,2,3,6,1)



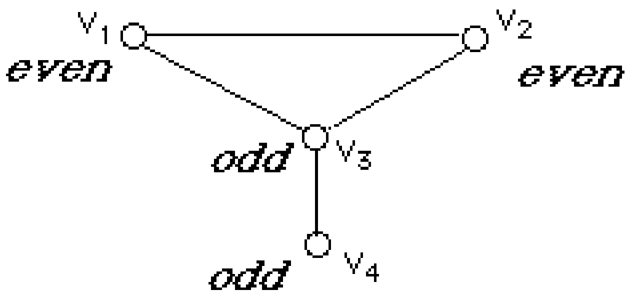
The DEGREE of a vertex is the number of edges incident with the vertex



vertex	*degree*
1	2
2	2
3	3
4	1

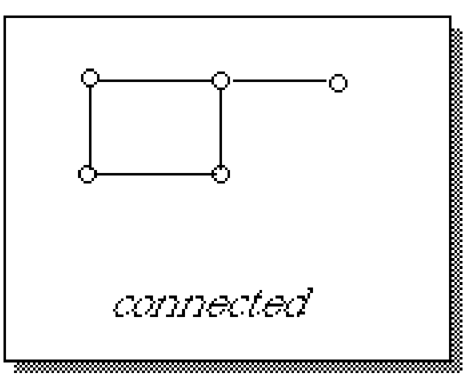
Theorem: The sum of the degrees of the vertices of a graph is twice the number of edges

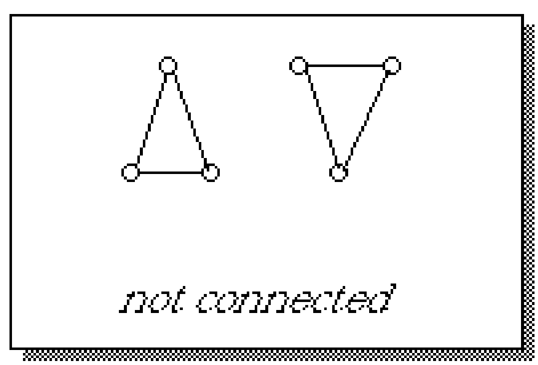
A vertex of a graph is **EVEN** or **ODD** according to whether its degree is an even or odd integer, respectively.



Theorem: Every graph contains an even number of odd vertices

A graph is **CONNECTED** if, for every pair of vertices, x & y, there is a chain of edges from vertex x to vertex y.

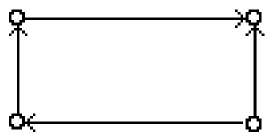




A directed graph is **CONNECTED**

if, for every pair of vertices, x & y, there is a chain of edges from vertex x to vertex y,

and **STRONGLY CONNECTED** if there is a path of edges from vertex x to vertex y.



connected, but not strongly connected

strongly connected

Suppose that we wish to assign directions to the edges of a connected graph so as to obtain a STRONGLY-CONNECTED digraph.

Under what conditions, if any, is this possible?

For example, can we make each street in a city one-way so that a vehicle at any intersection can reach any other intersection?

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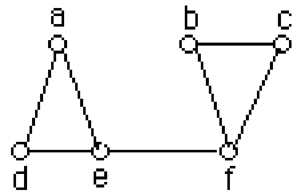
A BRIDGE of a connected graph is an edge which, if removed, destroys the graph's connectedness.



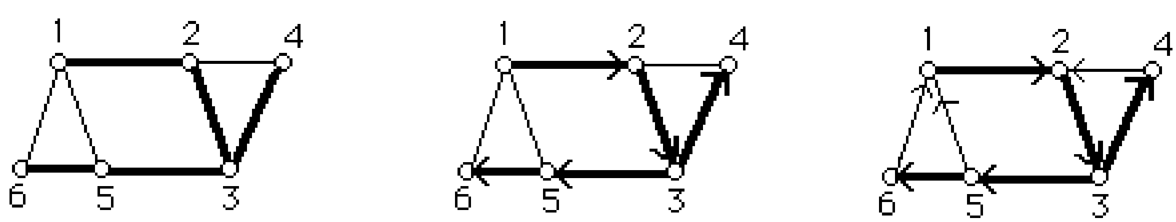
Edge (e,f) is a BRIDGE of the graph

Robbins' Theorem

A graph has a strongly-connected orientation
if and only if
the graph is connected and has no bridge.

Finding a Strongly-Connected Orientation

- First, find a DEPTH-FIRST-SEARCH SPANNING TREE
- Orient all edges ON the spanning tree from the vertex with smaller label to the vertex with the larger label
- Orient all edges NOT on the spanning tree from the vertex with larger label to the vertex with smaller label



DEPTH-FIRST-SEARCH SPANNING TREE

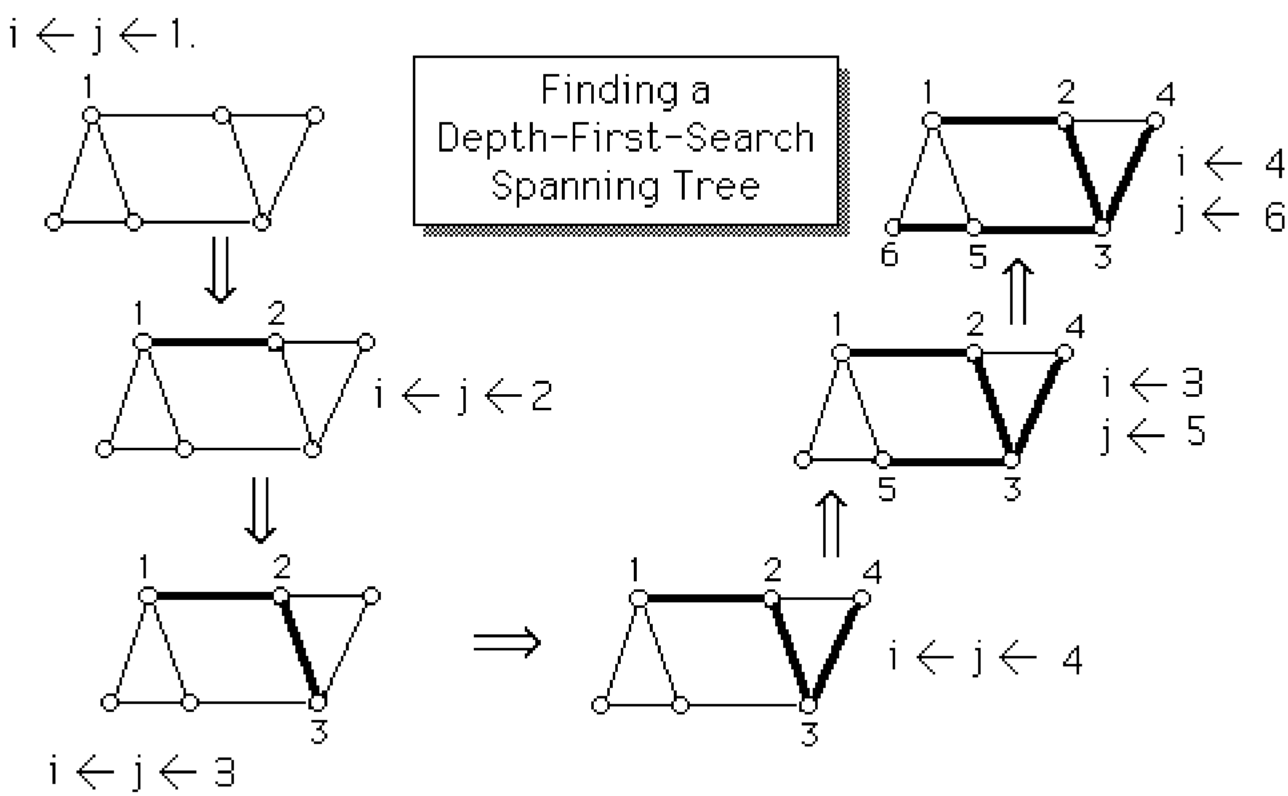
[0] Select any vertex, and label it "1". Let $i \leftarrow j \leftarrow 1$.

[1] Select any vertex which is connected by a single edge to the vertex labeled "i". If none, go to step [4]; otherwise, proceed to step [2]

[2] Label the selected vertex "j+1"

[3] Let $i \leftarrow j \leftarrow j+1$. Go to step [1].

[4] Let $i \leftarrow i-1$. If $i=0$, STOP; otherwise, go to step [1].



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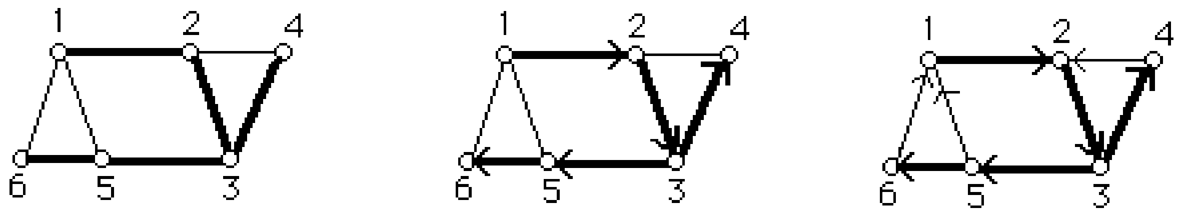
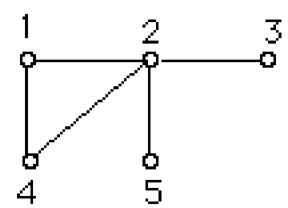
Example: Finding a strongly-connected orientation of a connected graph



depth-first-search spanning tree — *orient edges on the tree* — *orient edges not on the tree*

ADJACENCY MATRIX

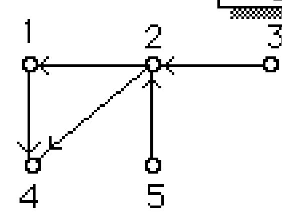
graph



$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$A_i^j = \begin{cases} 1 & \text{if there is an edge } (i,j) \\ 0 & \text{otherwise} \end{cases}$$

digraph

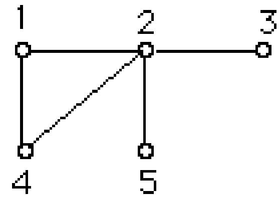


$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$A_i^j = \begin{cases} 1 & \text{if there is an arc } (i,j) \\ 0 & \text{otherwise} \end{cases}$$

REACHABILITY MATRIX

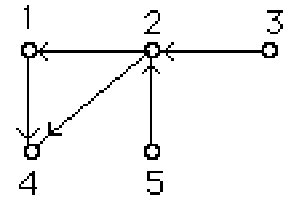
graph



$$R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$R_i^j = \begin{cases} 1 & \text{if there is a chain from vertex } i \text{ to vertex } j \\ 0 & \text{otherwise} \end{cases}$$

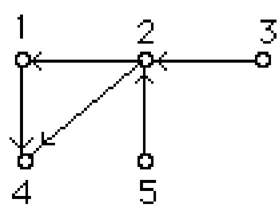
digraph



$$R = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$R_i^j = \begin{cases} 1 & \text{if there is a path from vertex } i \text{ to vertex } j \\ 0 & \text{otherwise} \end{cases}$$

Consider the generalized inner product $\vee . \wedge$ in APL notation: (*or*, *and*)



$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \end{bmatrix} \vee . \wedge \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \equiv (1 \wedge 1) \vee (0 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 0) \vee (0 \wedge 0)$$

$$\equiv 1 \vee 0 \vee 0 \vee 0 \vee 0$$

$$\equiv 1$$

row #2

column #4

indicates that there is an arc (2,1) ***and*** *an arc (1,4)*

indicates that there is a path of 2 arcs from 2 to 4

The value in row i & column j of the matrix

A ∨.∧ A

is 1 if there is a path, consisting of 2 arcs, from vertex i to vertex j, and 0 otherwise

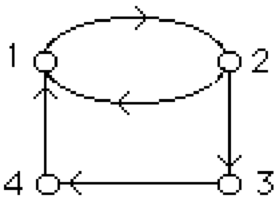
(A∨.∧A) ∨.∧ A has a 1 in row i&column j if there is a path consisting of 3 arcs from i to j etc.

How can the reachability matrix be computed?

An APL function to compute the reachability matrix:

```
    ∇R←A REACH N
[1]  →(N=0)/LAST
[2]  R ← A ∨.∧ A REACH N-1
[3]  →0
[4] LAST: R ← IDENTITY 1↑⍴A
    ∇
```

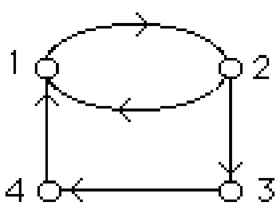
Powers of the Adjacency Matrix



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} +.\times \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

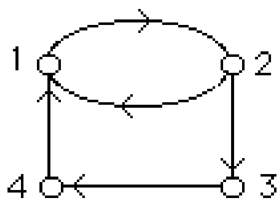
inner product (APL)



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} +.\times \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Theorem: If A is the adjacency matrix of a digraph, then the entry in row i & column j of A^k is the number of paths of length k edges from vertex i to vertex j



$$A^3 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad A^4 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$