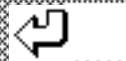


# Machine Replacement with Stochastic Failures



A component of a machine has an active life, measured in weeks, that is a random variable  $T$ , where

$$P\{T=1\} = 0.1, P\{T=2\} = 0.25, \\ P\{T=3\} = 0.35, P\{T=4\} = 0.3$$

Note that the component *never* survives more than 4 weeks.

Suppose that one starts with a fresh component.

At the beginning of each week, the component is inspected and is determined to be either operational or broken down.

*(That is, the component is not continuously monitored, and so the broken-down condition is only discovered at the beginning of the week.)*

At the beginning of the week, after determining the condition of the component, we may decide to replace it with a fresh component, or to continue with the current component.

*(Of course, if broken down, it must be replaced!)*

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The machine earns \$100 in revenues each week that it is operational with no breakdowns.

A replacement component costs \$50.

We wish to formulate a DP model to select a policy to maximize the machine's revenue over  $N$  weeks, i.e., to specify the age at which the component should be replaced.

We will assume here that at the end of the  $N$  weeks, there is no salvage value for an operational component, since the machine will be completely overhauled.

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**Stage**  $n$  = # weeks remaining in the planning period.

### State of system

$S_n$  = age of current component at end of stage  $n$ .  
 $S_n \in \{ 1, 2, 3, 4 \}$

*(We will consider state 4 to include the case in which the component has broken down, since these two states are indistinguishable.)*

### Decisions

$X_n = 0$     keep  
 $X_n = 1$     replace with a fresh component

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### Random outcome

$Z_n = 0$     component survives week  
           1    component fails

### Probability distribution

For each of the ages 1, 2, & 3, we need to compute the failure probability (conditional upon the component's having survived to that age and the decision being to keep the current component).

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For example,

$$p_{14}^0 = P\{Z_n=1 \mid S_n=1, X_n=0\} = P\{T=1 \mid T \geq 1\}$$

$$= \frac{P\{T=1\}}{\sum_{t=1}^4 P\{T=t\}} = \frac{0.1}{1} = 0.1 \leftarrow$$

$$p_{12}^0 = 1 - p_{14}^0 = 0.9$$

*Probability that the component fails during the next week, given that it is one week old.*

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$$p_{24}^0 = P\{Z_n=1 \mid S_n=2, X_n=0\}$$

$$= P\{T=2 \mid T \geq 2\}$$

$$= \frac{P\{T=2\}}{\sum_{t=2}^4 P\{T=t\}} = \frac{0.25}{0.25 + 0.35 + 0.3} = \frac{0.25}{0.9} = 0.27777$$

$$p_{23}^0 = 1 - p_{24}^0 = 0.72222$$

*etc.*

*Probability that the component fails during the next week, given that it is two weeks old.*

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Stochastic Machine Replacement Problem

State Vector

i	1	2	3	4
s(i)	1	2	3	4

Decision Vector

i	1	2
x(i)	0	1

Random Variable

i	1	2
d(i)	0	1

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Probability array

		survive	fail	
	z=0	1		
<b>Age</b>				
<b>s = 1</b>	x =	0	1	keep replace
		0.9	0.1	
		0.9	0.1	
<b>s = 2</b>	x =	0	1	keep replace
		0.72	0.28	
		0.9	0.1	
<b>s = 3</b>	x =	0	1	keep replace
		0.46	0.54	
		0.9	0.1	
<b>s = 4</b>	x =	0	1	keep
		0	1	
		0.9	0.1	

*P is a 3-dimensional array of conditional probabilities*

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# APL code

```

▽VALUE←F N;t;Return
[1]  ⍺
[2]  ⍺      Optimal Value Function for stochastic DP model
[3]  ⍺      of a machine replacement problem
[4]  ⍺
[5]  →LAST IF N=0
[6]  t←((s°.×(1-x))+1)°.+d   ⋄ t[;;2]←4
[7]  Return←((ρs)ρ0)°.+(-R_cost)°.+Revenue
[8]  VALUE←P MAXΔE Return + (F N-1)[TRANSITION t]
[9]  →0
[10] LAST:VALUE←((ρs)ρ0),-BIG
▽
    
```

R_cost	=	0	50
Revenue	=	100	0

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$$f_0(S_0) = 0 \quad \forall S_0$$

Stage 1

s	<i>keep</i> <i>replace</i>	
	x: 0	1
1	90.00	40.00
2	72.00	40.00
3	46.00	40.00
4	0.00	40.00

*Expected Revenues*

For example, if  $s=2$  and the machine is kept, there is a 72% probability that it will not fail, in which case the revenue is \$100, so the expected revenue is  $0.72(100)=72$

If the machine is replaced, there is a cost of \$50. There is a 90% probability that the replacement does not fail, so the expected revenue is  $0.9(100)-50 = 40$

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$$f_0(S_0) = 0 \quad \forall S_0$$

## Stage 1

		<i>keep</i>	<i>replace</i>
s	\ x:	0	1
1	90.00	40.00	
2	72.00	40.00	
3	46.00	40.00	
4	0.00	40.00	

*Expected Revenues*

*optimal policy:  
replace if broken-down (or age 4)*

S <sub>1</sub>	f <sub>1</sub> (S <sub>1</sub> )	Optimal Decisions
1	90.00	0
2	72.00	0
3	46.00	0
4	40.00	1

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S <sub>1</sub>	f <sub>1</sub> (S <sub>1</sub> )
1	90.00
2	72.00
3	46.00
4	40.00

	s	x: 0	1
1	158.80	125.00	
2	116.32	125.00	
3	86.00	125.00	
4	40.00	125.00	

## Stage 2

Using the optimal revenues from the final stage, i.e.,  $f_1(S_1)$ , we compute the expected revenues for each combination of state  $s$  and decision  $x$  at stage 2.

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Stage 2

If  $s=2$  and  $x=0$ ,

*(i.e., if component is two weeks old, and the decision is made to keep instead of replacing it.)*

expected revenue is

$$\begin{aligned}
 & P\{\text{failure}\} \left[ \begin{array}{c} \text{this week's} \\ \text{revenue} \end{array} + \begin{array}{c} \text{expected future} \\ \text{revenues} \end{array} \right] + P\{\text{survival}\} \left[ \begin{array}{c} \text{this week's} \\ \text{revenue} \end{array} + \begin{array}{c} \text{expected future} \\ \text{revenues} \end{array} \right] \\
 & 0.2777 ( 0 + f_1(4) ) + 0.7222 ( 100 + f_1(3) ) \\
 & = 0.2777 ( 0 + 40 ) + 0.7222 ( 100 + 46 ) \\
 & = 116.32
 \end{aligned}$$

s	x: 0	1
1	158.80	125.00
2	116.32	125.00
3	86.00	125.00
4	40.00	125.00

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$S_1$	$f_1(S_1)$
1	90.00
2	72.00
3	46.00
4	40.00



s	x: 0	1
1	158.80	125.00
2	116.32	125.00
3	86.00	125.00
4	40.00	125.00

Stage 2

*optimal policy:  
replace if age  
is 2 weeks or  
more*

$S_2$	$f_2(S_2)$	Optimal Decisions
1	158.80	0
2	125.00	1
3	125.00	1
4	125.00	1

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**Stage 3**

$S_2$	$f_2(S_2)$
1	158.80
2	125.00
3	125.00
4	125.00



$s$	$x: 0$	$1$
1	215.00	195.42
2	197.00	195.42
3	171.00	195.42
4	125.00	195.42

*optimal policy:  
replace if age  
is 3 weeks or  
more*

$S_3$	$f_3(S_3)$	Optimal Decisions
1	215.00	0
2	197.00	0
3	195.42	1
4	195.42	1

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**Stage 4**

$S_3$	$f_3(S_3)$
1	215.00
2	197.00
3	195.42
4	195.42



$s$	$x: 0$	$1$
1	286.84	253.04
2	267.42	253.04
3	241.42	253.04
4	195.42	253.04

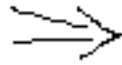
*optimal policy:  
replace if age  
is 3 weeks or  
more*

$S_4$	$f_4(S_4)$	Optimal Decisions
1	286.84	0
2	267.42	0
3	253.04	1
4	253.04	1

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Stage 5

$S_4$	$f_4(S_4)$
1	286.84
2	267.42
3	253.04
4	253.04



$s$	$x: 0$	$1$
1	355.98	323.46
2	325.04	323.46
3	299.04	323.46
4	253.04	323.46

*optimal policy:  
replace if age  
is 3 weeks or  
more*

$S_5$	$f_5(S_5)$	Optimal Decisions
1	355.98	0
2	325.04	0
3	323.46	1
4	323.46	1

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Stage 6

$S_5$	$f_5(S_5)$
1	355.98
2	325.04
3	323.46
4	323.46



$s$	$x: 0$	$1$
1	414.88	392.73
2	395.46	392.73
3	369.46	392.73
4	323.46	392.73



*optimal policy:  
replace if age  
is 3 weeks or  
more*

$S_6$	$f_6(S_6)$	Optimal Decisions
1	414.88	0
2	395.46	0
3	392.73	1
4	392.73	1

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The total expected revenue if we have a week-old component at stage 6, is \$414.88

The optimal policy for all stages except 1 & 2 (the final 2 stages) is to replace only if the component is age  $\geq 3$  (or broken-down).

