Machine Replacement with Stochastic Failures

author

A component of a machine has an active life, measured in weeks, that is a random variable T, where

$$P\{T=1\} = 0.1,\ P\{T=2\} = 0.25,$$
$$P\{T=3\} = 0.35,\ P\{T=4\} = 0.3$$

Note that the component *never* survives more than 4 weeks.

Suppose that one starts with a fresh component.

At the beginning of each week, the component is inspected and is determined to be either operational or broken down.

(That is, the component is not continuously monitored, and so the broken-down condition is only discovered at the beginning of the week.)

At the beginning of the week, after determining the condition of the component, we may decide to replace it with a fresh component, or to continue with the current component.

(Of course, if broken down, it must be replaced!)

The machine earns \$100 in revenues each week that it is operational with no breakdowns.

A replacement component costs \$50.

We wish to formulate a DP model to select a policy to maximize the machine's revenue over N weeks, i.e., to specify the age at which the component should be replaced.

We will assume here that at the end of the N weeks, there is no salvage value for an operational component, since the machine will be completely overhauled.

Stage n = # weeks remaining in the planning period.

State of system

S_n = age of current component at end of stage n.

$$S_n \in \{1, 2, 3, 4\}$$

(We will consider state 4 to include the case in which the component has broken down, since these two states are indistinguishable.)

Decisions

$X_n = 0$ keep

$X_n = 1$ replace with a fresh component

Random outcome

$Z_n =$ 0 component survives week

1 component fails

Probability distribution

For each of the ages 1, 2, & 3, we need to compute the failure probability (conditional upon the component's having survived to that age and the decision being to keep the current component).

For example,

$$p^0_{14} = P\{Z_n=1 \mid S_n=1, X_n=0\} = P\{T=1 \mid T\geq 1\}$$

$$= \frac{P\{T=1\}}{\sum_{t=1}^{4} P\{T=t\}} = \frac{0.1}{1} = 0.1$$

Probability that the component fails during the next week, given that it is one week old.

$$p^0_{12} = 1 - p^0_{14} = 0.9$$

$$p^0_{24} = P\{Z_n=1 \mid S_n=2, X_n=0\}$$

$$= P\{T=2 \mid T\geq 2\}$$

$$= \frac{P\{T=2\}}{\sum_{t=2}^{4} P\{T=t\}} = \frac{0.25}{0.25+0.35+0.3} = \frac{0.25}{0.9} = 0.27777$$

Probability that the component fails during the next week, given that it is two weeks old.

$$p^0_{23} = 1 - p^0_{24} = 0.72222$$

etc.

Stochastic Machine Replacement Problem

State Vector

i	1	2	3	4
s[i]	1	2	3	4

Decision Vector

i	1	2
x[i]	0	1

Random Variable

i	1	2
d[i]	0	1

Probability array

Age	x	survive z=0	fail 1	
s = 1	x = 0	0.9	0.1	keep
	x = 1	0.9	0.1	replace
s = 2	x = 0	0.72	0.28	keep
	x = 1	0.9	0.1	replace
s = 3	x = 0	0.46	0.54	keep
	x = 1	0.9	0.1	replace
s = 4	x = 0	0	1	keep
	x = 1	0.9	0.1	

P is a 3-dimensional array of conditional probabilities

APL code

```
    ∇VALUE←F N;t;Return
      ⍝
      ⍝      Optimal Value Function for stochastic DP model
      ⍝      of a machine replacement problem
      ⍝
      →LAST IF N=0
      t←((s∘.×(1-x))+1)∘.+d   ⋄ t[;;2]←4
      Return←((⍴s)⍴0)∘.+(-R_cost)∘.+Revenue
      VALUE←P MAX∆E Return + (F N-1)[TRANSITION t]
      →0
      LAST:VALUE←((⍴s)⍴0),-BIG
    ∇
```

```
R_cost  =    0  50
Revenue =  100   0
```


Stage 1

$f_0(S_0) = 0 \;\; \forall S_0$

	keep	*replace*
s \ x:	0	1
1	90.00	40.00
2	72.00	40.00
3	46.00	40.00
4	0.00	40.00

Expected Revenues

For example, if s=2 and the machine is kept, there is a 72% probability that it will not fail, in which case the revenue is \$100, so the expected revenue is 0.72(100)=72

If the machine is replaced, there is a cost of \$50. There is a 90% probability that the replacement does not fail, so the expected revenue is 0.9(100)-50 = 40

$f_0(S_0) = 0 \quad \forall S_0$

Stage 1

s \ x:	0 (keep)	1 (replace)
1	90.00	40.00
2	72.00	40.00
3	46.00	40.00
4	0.00	40.00

Expected Revenues

optimal policy: replace if broken-down (or age 4)

S_1	$f_1(S_1)$	Optimal Decisions
1	90.00	0
2	72.00	0
3	46.00	0
4	40.00	1

Stage 2

S_1	$f_1(S_1)$
1	90.00
2	72.00
3	46.00
4	40.00

s	x: 0	1
1	158.80	125.00
2	116.32	125.00
3	86.00	125.00
4	40.00	125.00

Using the optimal revenues from the final stage, i.e., $f_1(S_1)$, we compute the expected revenues for each combination of state s and decision x at stage 2.

Stage 2

If $s=2$ and $x=0$, *(i.e., if component is two weeks old, and the decision is made to keep instead of replacing it.)*

expected revenue is

$$P\{\text{failure}\}\left[\text{this week's revenue} + \text{expected future revenues}\right] + P\{\text{survival}\}\left[\text{this week's revenue} + \text{expected future revenues}\right]$$

$$0.2777(\ 0 + f_1(4)\ ) + 0.7222(\ 100 + f_1(3))$$

$$= 0.2777(\ 0 + 40) + 0.7222(100 + 46)$$

$$= 116.32$$

s	x: 0	1
1	158.80	125.00
2	116.32	125.00
3	86.00	125.00
4	40.00	125.00

Stage 2

S_1	$f_1(S_1)$
1	90.00
2	72.00
3	46.00
4	40.00

s	x: 0	1
1	158.80	125.00
2	116.32	125.00
3	86.00	125.00
4	40.00	125.00

optimal policy: replace if age is 2 weeks or more

S_2	$f_2(S_2)$	Optimal Decisions
1	158.80	0
2	125.00	1
3	125.00	1
4	125.00	1

Stage 3

S_2	$f_2(S_2)$
1	158.80
2	125.00
3	125.00
4	125.00

→

s	x: 0	1
1	215.00	195.42
2	197.00	195.42
3	171.00	195.42
4	125.00	195.42

optimal policy: replace if age is 3 weeks or more

S_3	$f_3(S_3)$	Optimal Decisions
1	215.00	0
2	197.00	0
3	195.42	1
4	195.42	1

Stage 4

S_3	$f_3(S_3)$
1	215.00
2	197.00
3	195.42
4	195.42

→

s	x: 0	1
1	286.84	253.04
2	267.42	253.04
3	241.42	253.04
4	195.42	253.04

optimal policy: replace if age is 3 weeks or more

S_4	$f_4(S_4)$	Optimal Decisions
1	286.84	0
2	267.42	0
3	253.04	1
4	253.04	1

Stage 5

S_4	$f_4(S_4)$
1	286.84
2	267.42
3	253.04
4	253.04

⟹

s	x: 0	1
1	355.98	323.46
2	325.04	323.46
3	299.04	323.46
4	253.04	323.46

optimal policy: replace if age is 3 weeks or more

S_5	$f_5(S_5)$	Optimal Decisions
1	355.98	0
2	325.04	0
3	323.46	1
4	323.46	1

Stage 6

S_5	$f_5(S_5)$
1	355.98
2	325.04
3	323.46
4	323.46

⟹

s \ x:	0	1
1	414.88	392.73
2	395.46	392.73
3	369.46	392.73
4	323.46	392.73

⇓

optimal policy: replace if age is 3 weeks or more

S_6	$f_6(S_6)$	Optimal Decisions
1	414.88	0
2	395.46	0
3	392.73	1
4	392.73	1

The total expected revenue if we have a week-old component at stage 6, is $414.88

The optimal policy for all stages except 1 & 2 (the final 2 stages) is to replace only if the component is age ≥ 3 (or broken-down).