Disjoint Path Problem

an application of Lagrangian Relaxation

author

Application

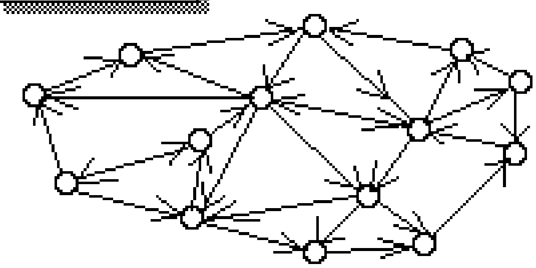
A set of products is to be scheduled on a machine.
(Example: scheduling steel to be rolled (producing varying grades, widths, thicknesses, etc.) in a hot strip mill.)

For some pairs (i,j) of products, no major setup is required if product j immediately follows product i.

We wish to sequence the products so as to minimize the number of major setups required.

Represent the products by nodes in a network, with arc from node i to node j if node j requires no major setup when if follows node i.

Example



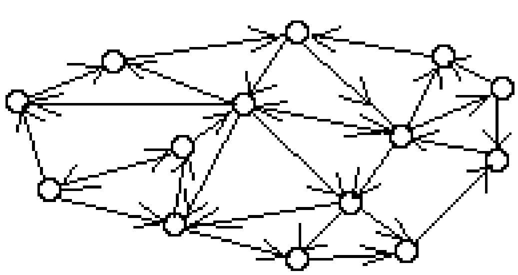
The nodes on a path through the network correspond to a sequence of products which can be produced with a single major setup.

Any two such paths should be *disjoint*, i.e., should share no common products.

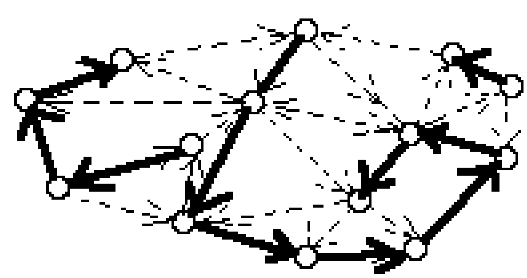
The Disjoint Path Problem:

Find the minimum number of disjoint paths which span all the nodes of a directed graph.

Example



A feasible solution



(3 paths!)

PROBLEM STATEMENT:

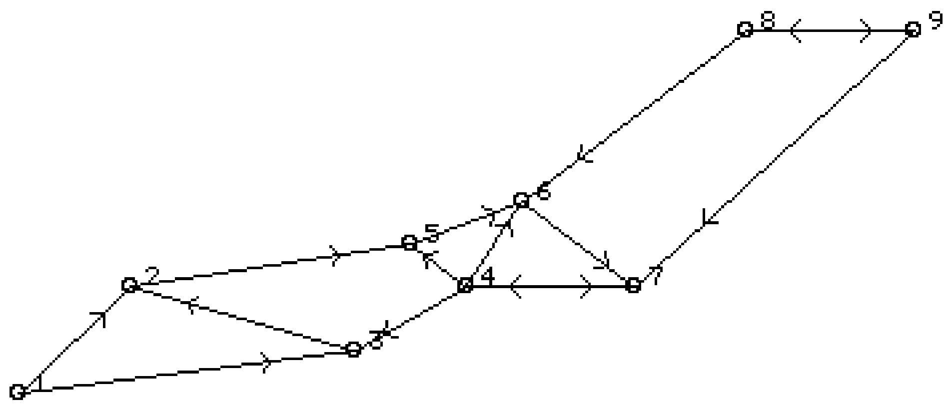
Given a directed graph (digraph) $G = (N,A)$

where $N = \{1, 2, \ldots n\}$ = set of nodes

A = set of arcs ($A \subseteq N \times N$)

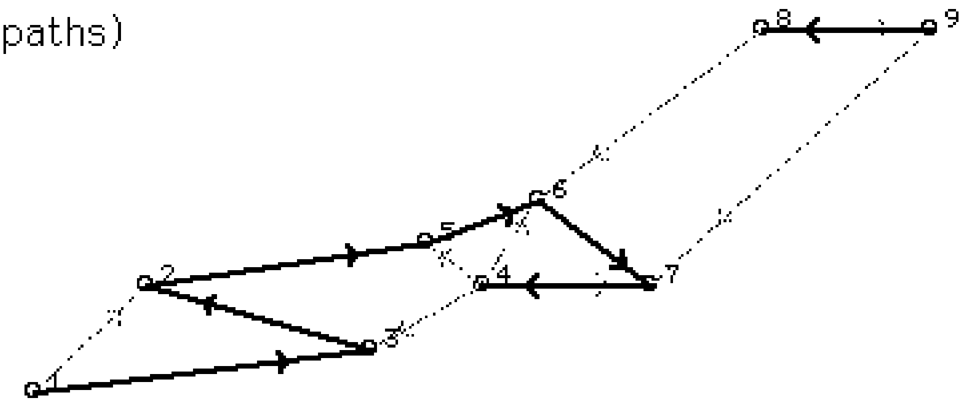
Find the minimum number of paths such that
every node $i \in N$ lies on one (and only one) path

Example:



The optimal solution:

(2 paths)



Mathematical Programming Model

Define the variables

$$X_{ij} = \begin{cases} 1 & \text{if arc } (i,j) \text{ is included on a path} \\ 0 & \text{otherwise} \end{cases}$$

Clearly $X_{ij} = 1$ for at most one j for each i
and $X_{ij} = 1$ for at most one i for each j

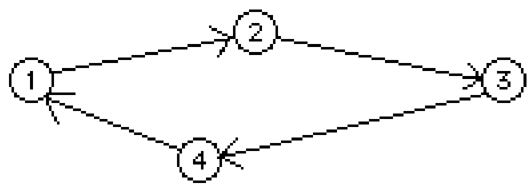
That is, at most one arc enters node j,
and at most one arc leaves node i

Thus, we have the constraints

$$\sum_{j=1}^{n} X_{ij} \le 1 \quad \textbf{for each } i \varepsilon N$$

$$\sum_{i=1}^{n} X_{ij} \le 1 \quad \textbf{for each } j \varepsilon N$$

However, the above constraints permit circuits, e.g.,



We must add the constraint that the edges of the subgraph indicated by X form a "forest", i.e., a collection of trees.

(A tree is a subgraph containing no cycle.)

In order to facilitate defining the objective function (which is to be the number of paths) in terms of X,

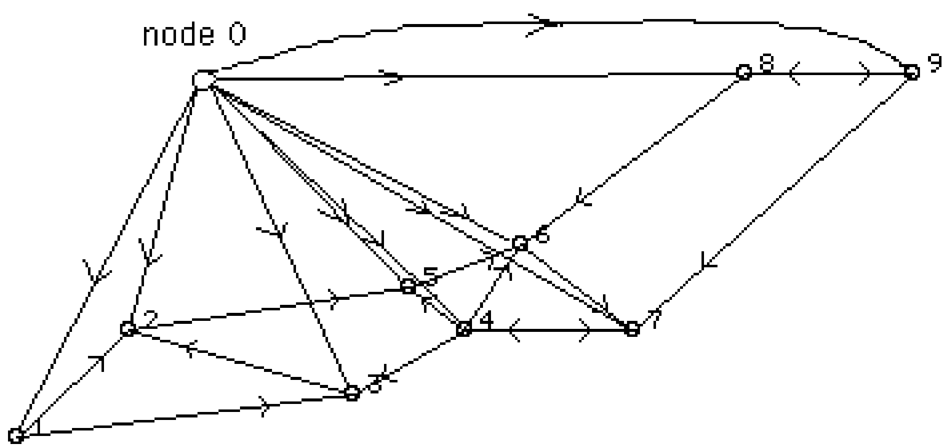
Define a new node 0

Let $G' = (N', A')$ where

$$N' = N \cup \{0\}$$

$$A' = A \cup \{(0,1), (0,2), \dots (0,n)\}$$

Let $X_{0i} = \begin{cases} 1 & \text{if node } i \text{ is the beginning of a path} \\ 0 & \text{otherwise} \end{cases}$



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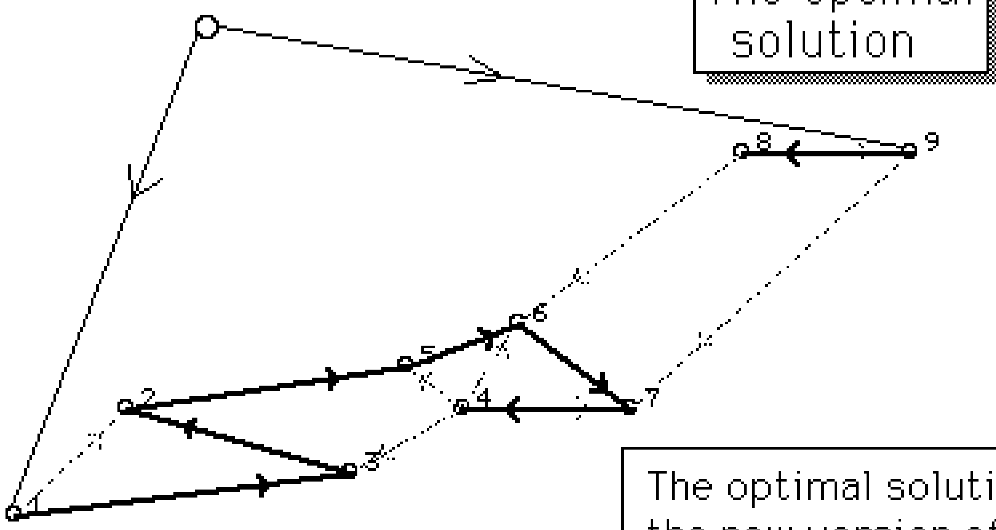


The Optimization Problem:

Minimize $\sum_{j=1}^{n} X_{0j}$

subject to

$X \in \mathcal{T}$ = **set of all spanning trees of G'**

$\sum_{j=1}^{n} X_{ij} \leq 1$ **for each** $i \in N$

$\sum_{i=0}^{n} X_{ij} = 1$ **for each** $j \in N$

$X_{ij} \in \{0,1\}$ **for each** $(i,j) \in A'$

Note that no inequality limits out-degree of node 0

Minimize $\sum_{j=1}^{n} X_{0j}$

subject to

$X \in \mathcal{T}$ = set of all spanning trees of G'

$\sum_{j=1}^{n} X_{ij} \leq 1$ for each $i \in N$

$\sum_{i=0}^{n} X_{ij} = 1$ for each $j \in N$

These are essentially constraints of an assignment problem!

$X_{ij} \in \{0,1\}$ for each $(i,j) \in A'$

This problem appears to be a good candidate for Lagrangian Relaxation because of its structure:

- If we relax the spanning tree constraint, we obtain a relaxation which is an assignment problem
- If we relax the assignment constraints, we obtain a relaxation which is a minimum spanning tree problem

However, because the spanning tree constraint is not easily written as a system of explicit linear constraints, relaxing them is problematic!

Variable "splitting"

For each variable X_{ij} of the problem, define a variable Y_{ij}

Require that X be a spanning tree,
that Y be a feasible assignment,
and that $X_{ij} = Y_{ij}$ for each i & j

Minimize $\alpha \sum_{j=1}^{n} X_{0j} + (1-\alpha) \sum_{j=1}^{n} Y_{0j}$

subject to

$X \in \mathcal{T}$

$$\sum_{j=1}^{n} Y_{ij} \leq 1 \quad \textbf{for each } i \in N$$

$$\sum_{i=0}^{n} Y_{ij} = 1 \quad \textbf{for each } j \in N$$

$$Y_{ij} \in \{0,1\} \quad \textbf{for each } (i,j) \in A'$$

$$X_{ij} = Y_{ij} \quad \textbf{for each } (i,j) \in A'$$

for some specified weight α which distributes the cost between the two sets of variables $(0 \leq \alpha \leq 1)$

$$\text{Minimize } \alpha \sum_{j=1}^{n} X_{0j} + (1-\alpha) \sum_{j=1}^{n} Y_{0j}$$

subject to

$X \;\varepsilon\; \mathcal{T}$

$$\sum_{j=1}^{n} Y_{ij} \le 1 \quad \text{for each } i \varepsilon N$$

$$\sum_{i=0}^{n} Y_{ij} = 1 \quad \text{for each } j \varepsilon N$$

$$Y_{ij} \;\varepsilon\; \{0,1\} \quad \text{for each } (i,j)\, \varepsilon\, A$$

$$X_{ij} = Y_{ij} \quad \text{for each } (i,j)\, \varepsilon\, A'$$

We now relax these constraints!

The Lagrangian Relaxation:

$$\text{Minimize } \alpha \sum_{j=1}^{n} X_{0j} + (1-\alpha) \sum_{j=1}^{n} Y_{0j} + \sum_{i=0}^{n} \sum_{j=1}^{n} \lambda_{ij} (X_{ij} - Y_{ij})$$

subject to

$X \;\varepsilon\; \mathcal{T}$

$$\sum_{j=1}^{n} Y_{ij} \le 1 \quad \text{for each } i \varepsilon N$$

$$\sum_{i=0}^{n} Y_{ij} = 1 \quad \text{for each } j \varepsilon N$$

$$Y_{ij} \;\varepsilon\; \{0,1\} \quad \text{for each } (i,j)\, \varepsilon\, A'$$

The Lagrangian Relaxation:

Minimize $\sum_{j=1}^{n}(\alpha+\lambda_{0j})X_{0j}+\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{ij}X_{ij}$

$$+\sum_{j=1}^{n}(1-\alpha-\lambda_{0j})Y_{0j}-\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{ij}Y_{ij}$$

subject to

$X \in \mathcal{T}$

$\sum_{j=1}^{n} Y_{ij} \le 1$ **for each** $i \in N$

$\sum_{i=0}^{n} Y_{ij} = 1$ **for each** $j \in N$

$Y_{ij} \in \{0,1\}$ **for each** $(i,j) \in A'$

The Lagrangian Relaxation separates into two subproblems:

Minimum Spanning Tree Problem:

$\Phi_X(\lambda) =$ **minimum** $\sum_{j=1}^{n}(\alpha+\lambda_{0j})X_{0j}+\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{ij}X_{ij}$

subject to

$X \in \mathcal{T}$

Assignment Problem

$$\Phi_Y(\lambda) = \text{minimum} \sum_{j=1}^{n} (1 - \alpha - \lambda_{0j}) Y_{0j} - \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij} Y_{ij}$$

subject to

$$\sum_{j=1}^{n} Y_{ij} \leq 1 \quad \text{for each } i \in N$$

$$\sum_{i=0}^{n} Y_{ij} = 1 \quad \text{for each } j \in N$$

$$Y_{ij} \in \{0,1\} \quad \text{for each } (i,j) \in A'$$

For any matrix λ of Lagrangian multipliers. the sum of the optimal values of the two subproblems provides a lower bound on the optimal value of the original problem:

$$\Phi(\lambda) = \Phi_X(\lambda) + \Phi_Y(\lambda) \leq Z^*$$

The Lagrangian Dual:

$$\Phi^* = \text{Maximum } \Phi(\lambda)$$

The search for the optimal dual variables (λ) can be performed by ***subgradient optimization***

The subgradient of the dual objective, $\Phi(\lambda)$ is the matrix $\Delta = \{ \delta_{ij} \}$ where $\delta_{ij} = (X_{ij} - Y_{ij})$

This is the direction in which to change λ

$$\lambda_{new} = \lambda_{old} + \tau \frac{\left(Z^{*} - \Phi(\lambda_{old})\right)}{\left\|\Delta\right\|^{2}} \Delta$$

$\tau \in (0,2]$ *is a stepsize parameter*

It may be that the optimal values of X and Y for the subproblems are never feasible paths.

For this reason, it is worthwhile to seek a feasible solution (which provides an upper bound) by means of a heuristic.

Two heuristic algorithms have been designed:

- a "greedy" algorithm
- a random-search algorithm

The "greedy" algorithm proceeds as follows:

Initially, the path set P is empty ($P \leftarrow \emptyset$)

(a) If all nodes lie on a path, stop. Else, begin a new path by selecting the node i* which minimizes λ_{0i}.
Let $P \leftarrow P \cup \{(0,i^*)\}$

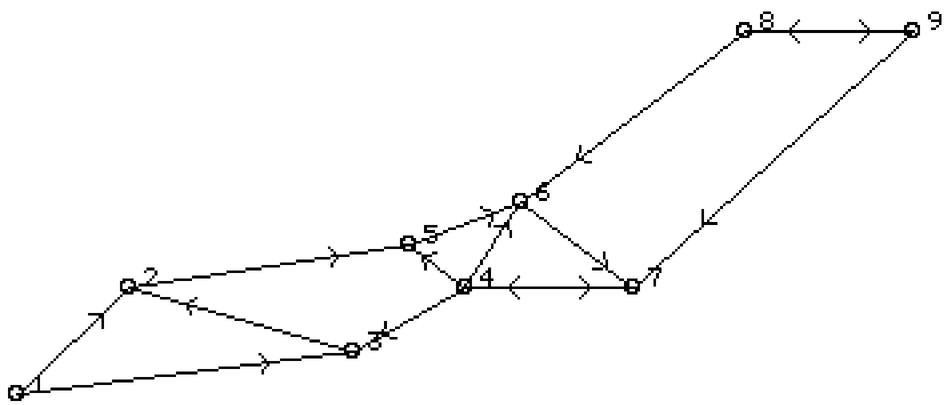
(b) If $\{(i,j) : j$ does not lie on a path$\}$ is empty, go to step (a).
Otherwise, let $j^* \leftarrow \text{argmin}\{\lambda_{ij} : j \text{ does not lie on a path}\}$

(c) Let $P \leftarrow P \cup \{(i^*,j^*)\}$ and $i^* \leftarrow j^*$.
Return to step (b).

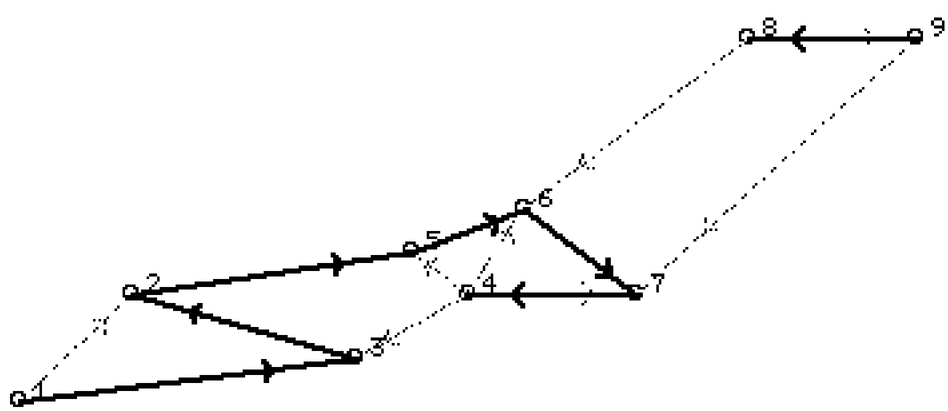
The random search algorithm finds several trial solutions, each constructed as in the greedy algorithm except:

In step (b), the choice of the next node to add to the path is random, with probability depending upon the current value of the Lagrange multipliers (λ_{ij}). *(Probabilities vary inversely as the multipliers, so that the choice tends to be "greedy".)*

Randomly-generated problem (N=9)

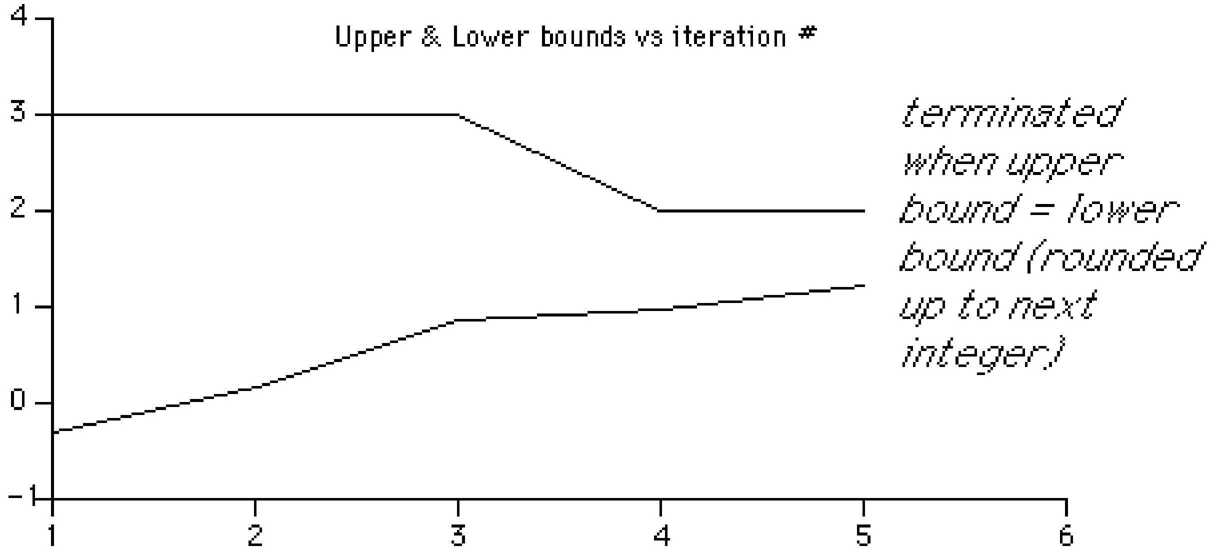


The optimal solution:



Results of Lagrangian dual search

(Spanning tree & assignment subproblems)

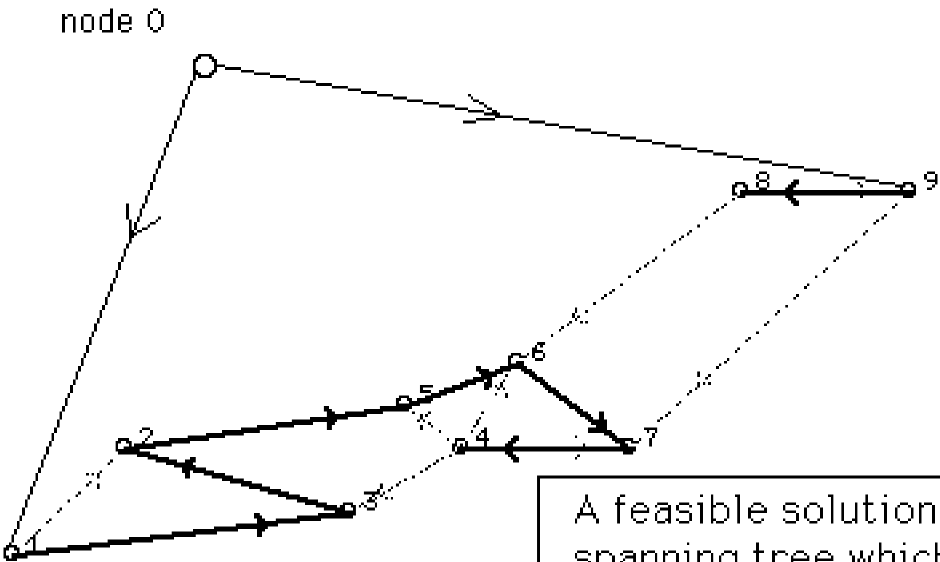


Other relaxations are possible:

#2 Relax, in addition to those relaxed in the approach just presented, the constraint on the in-degree of each node:

$$\sum_{i=0}^{n} Y_{ij} = 1 \quad \text{for each } j \in N$$

The subproblem in Y is then a simple GUB (generalized upper bound, or "multiple choice") problem.



A feasible solution is a directed spanning tree which is known as an "arborescence": the number of arcs entering any node is at most 1

#3

Replace the constraint that X is a tree with the stronger constraint that X is an "arborescence" (a directed tree with in-degrees of the nodes ≤ 1.) Then relax as in #2.

(The algorithm to compute a minimum spanning arborescence is $O(n^4)$. In practice, execution time for the APL code is about 15 times that for the spanning tree problem, for a 20-node problem.)

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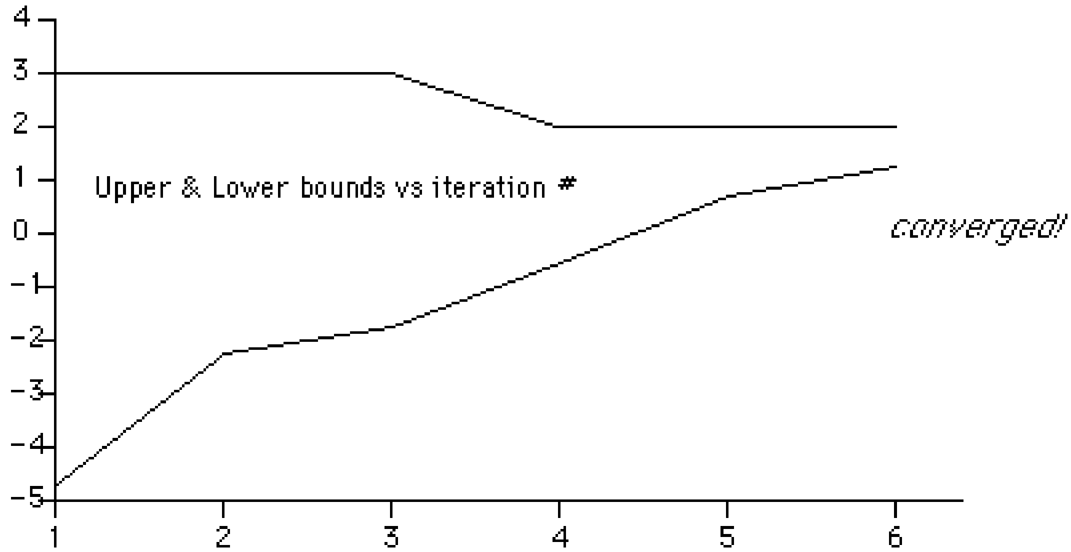
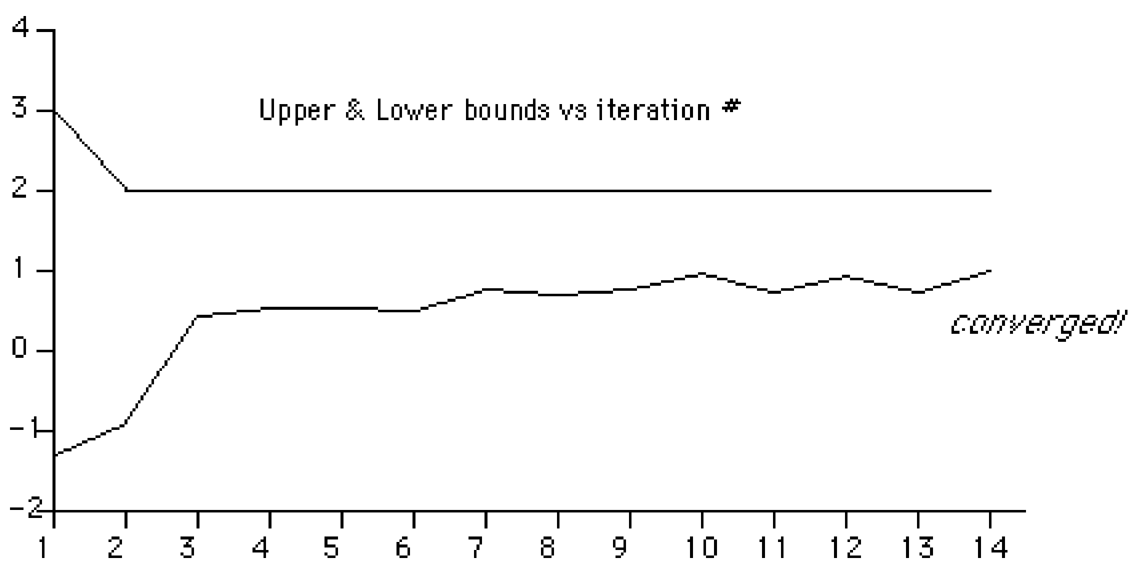
Using relaxation #2 (spanning tree & GUB problems)

(Using greedy heuristic)

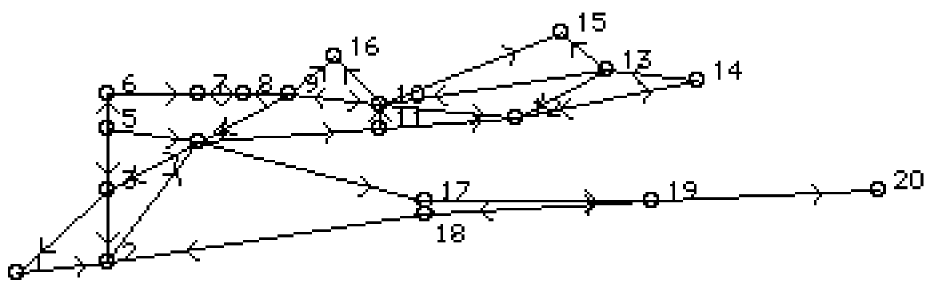


Using relaxation #3 (spanning arborescence & GUB)

(Using greedy heuristic)



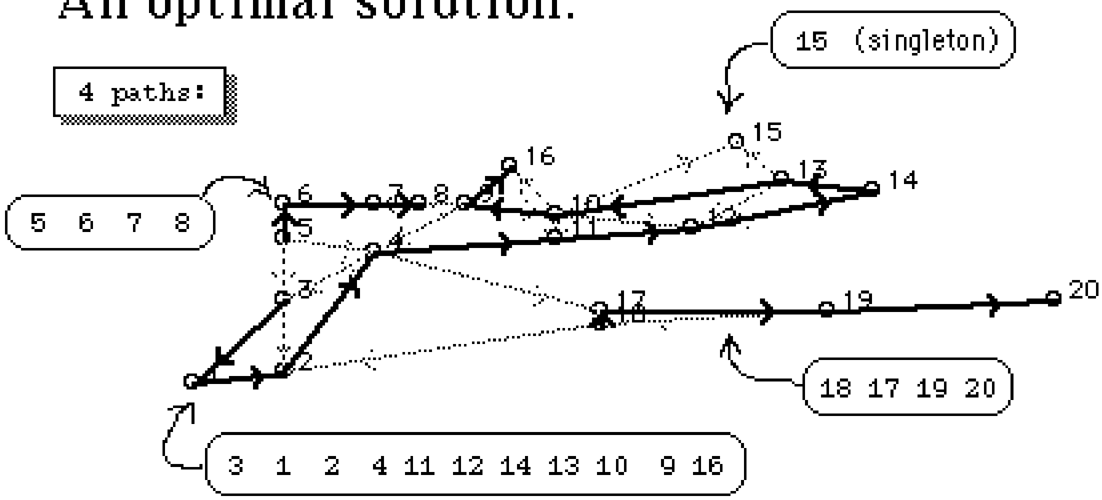
Another randomly-generated problem, with N=20



The Adjacency Matrix:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0
5	0	0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	1	0	0	0	1	0	1	0	0	0	0	0	1	0	0	0	0
10	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1	1	0	0	0	0
11	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	1	0	1	0	0	1	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0
18	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

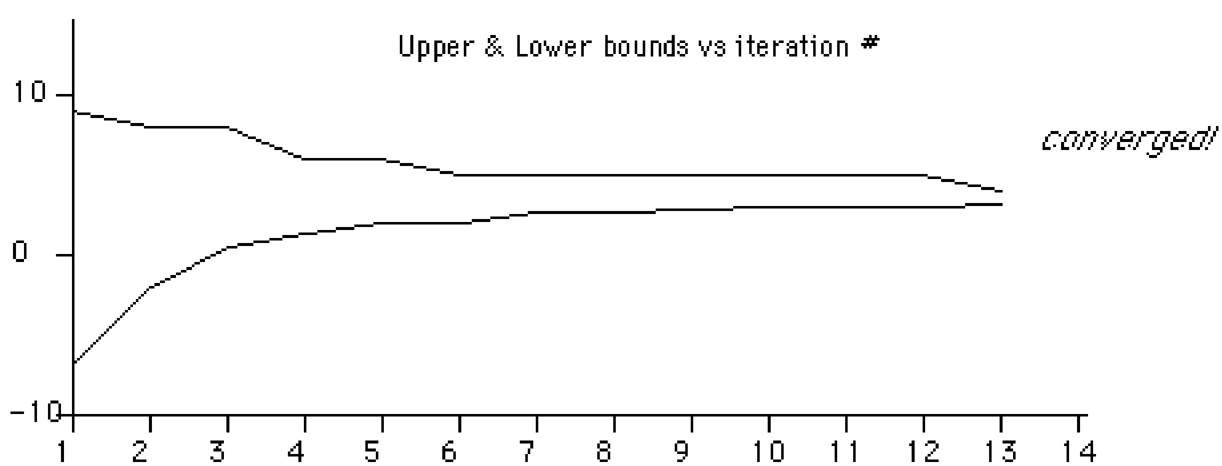
An optimal solution:



(The "dummy" node 0 & arcs from it are not shown.)

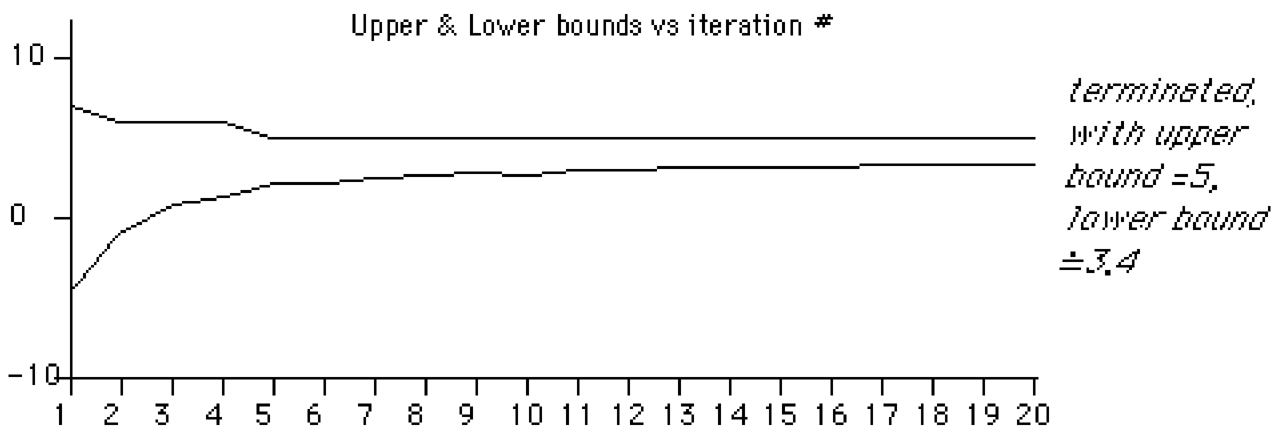
Relaxation #2 (spanning tree & GUB subproblems)

(Using random search heuristic with 5 trials.)



Relaxation #3 (spanning arborescence & GUB subproblems)

(Using random search heuristic with 5 trials.)



This limited computational experience suggests that the additional effort required to find the minimum spanning arborescence is not effective.