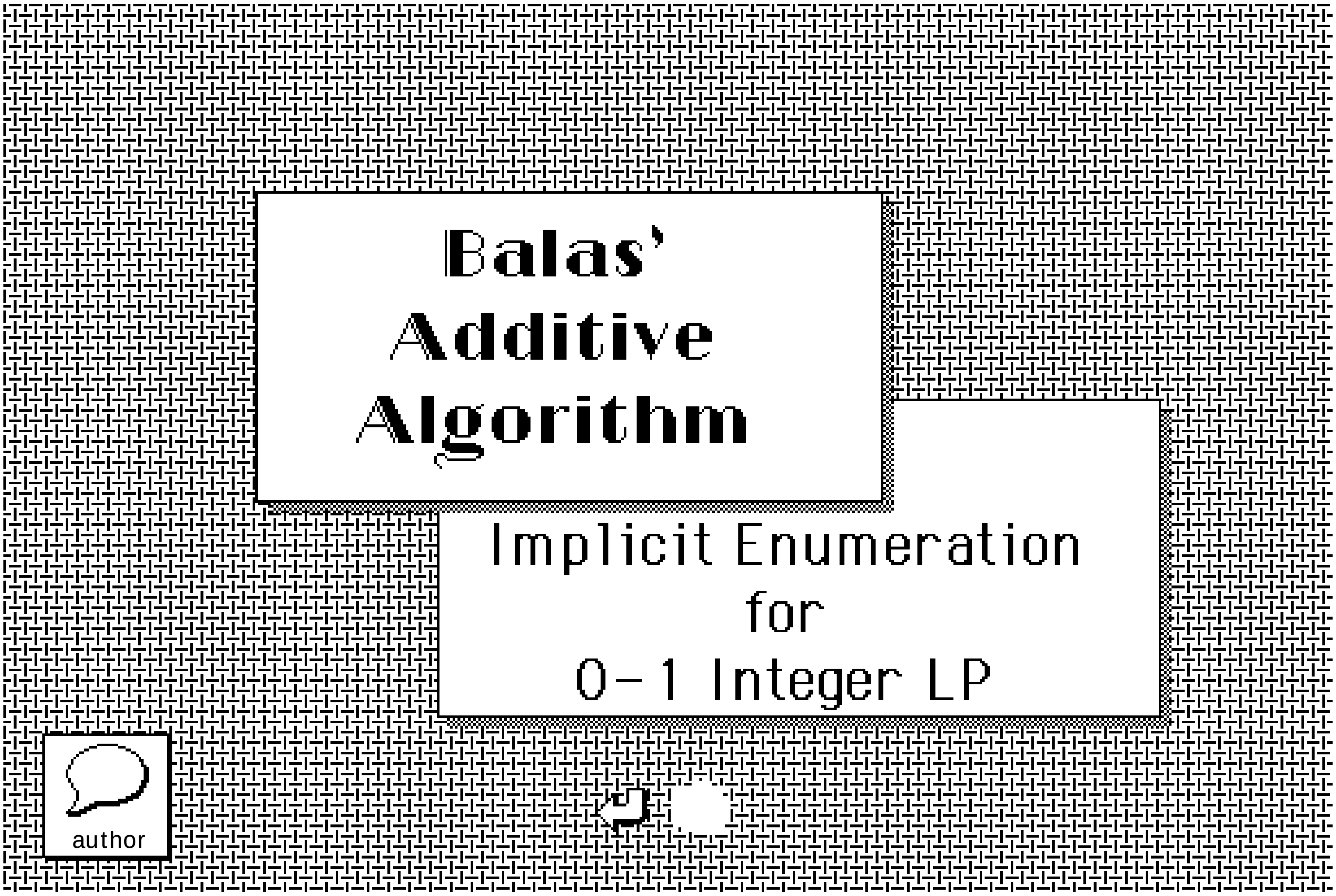



Egon Balas' algorithm for optimally solving zero-one LP problems is often referred to as...

**Implicit Enumeration**

and, because it requires only addition & subtraction (no multiplication or divisions),

**Additive Algorithm**

- Standard Form of Problem
- Explicit & Implicit Enumeration
- Partial Solutions & Completions
- Fathoming Tests

Examples

- One
- Two
- Three

## Standard Form

Let's assume that the problem is of the form:

$$\text{Minimize } z = \sum_{j \in N} C_j X_j$$

$$\text{subject to } \sum_{j \in N} a_{ij} X_j \le b_i, \ \forall i \in M$$

$$X_j \in \{0,1\}, \ \forall j \in N$$

where $M = \{1,2,3,\ldots,m\}$ and $N = \{1,2,3,\ldots,n\}$

and $C_j \geq 0 \ \forall j \in N$

*nonnegative costs!*

## Example

Maximize $-2X_1 + X_2 - 3X_3 + X_4$
subject to

$$X_1 + 2X_2 - X_3 \geq 1$$
$$-2X_1 + X_2 - X_4 \leq 3$$
$$X_j \in \{0,1\},\ j=1,2,3,4$$

*NOT in standard form...*

*objective is maximize, not minimize*
*costs differ in sign*
*one constraint is "greater-than-or-equal"*

*Replace "Max z" with "– Min –z"*
*and "≥" with "≤"*

\- Minimize $2X_1 - X_2 + 3X_3 - X_4$
subject to

$$-X_1 - 2X_2 + X_3 \leq -1$$
$$-2X_1 + X_2 - X_4 \leq 3$$
$$X_j \in \{0,1\},\ j=1,2,3,4$$

*For each variable* $X_j$ *having a negative cost, substitute* $1 - Y_j$ *where* $Y_j \in \{0,1\}$ *is the complement of* $X_j$.

- Minimize $2X_1 - (1-Y_2) + 3X_3 - (1-Y_4)$

subject to

$$-X_1 - 2(1-Y_2) + X_3 \leq -1$$
$$-2X_1 + (1-Y_2) - (1-Y_4) \leq 3$$
$$X_j \in \{0,1\},\ j=1,3$$
$$Y_j \in \{0,1\},\ j=2,4$$

*That is, the original problem is equivalent to the following problem, which is in the "standard form" for Balas' algorithm:*

2 - Minimize $2X_1 + Y_2 + 3X_3 + Y_4$

subject to

$$-X_1 + Y_2 + X_3 \leq 1$$
$$-2X_1 - Y_2 + Y_4 \leq 2$$
$$X_j \in \{0,1\},\ j=1,3$$
$$Y_j \in \{0,1\},\ j=2,4$$

# Example

$$\begin{array}{ll} \text{Minimize} & 3X_1 + 8X_2 + X_3 + 16X_4 + X_5 \\ \text{subject to} & X_1 - 2X_2 - 6X_3 + 2X_4 + 3X_5 \le 0 \\ & X_1 - 3X_3 - 2X_4 + 2X_5 \le -2 \\ & X_1 - 5X_2 + 4X_3 - X_4 - 2X_5 \le -5 \\ & X_j \in \{0,1\},\ j=1,2,3,4,5 \end{array}$$

There are $2^5 = 32$ binary vectors of length 5, which we could explicitly enumerate.

$$\begin{array}{ll} \text{Minimize} & 3X_1 + 8X_2 + X_3 + 16X_4 + X_5 \\ \text{subject to} & X_1 - 2X_2 - 6X_3 + 2X_4 + 3X_5 \le 0 \\ & X_1 - 3X_3 - 2X_4 + 2X_5 \le -2 \\ & X_1 - 5X_2 + 4X_3 - X_4 - 2X_5 \le -5 \\ & X_j \in \{0,1\},\ j=1,2,3,4,5 \end{array}$$

For each of the 32 binary vectors, let's evaluate

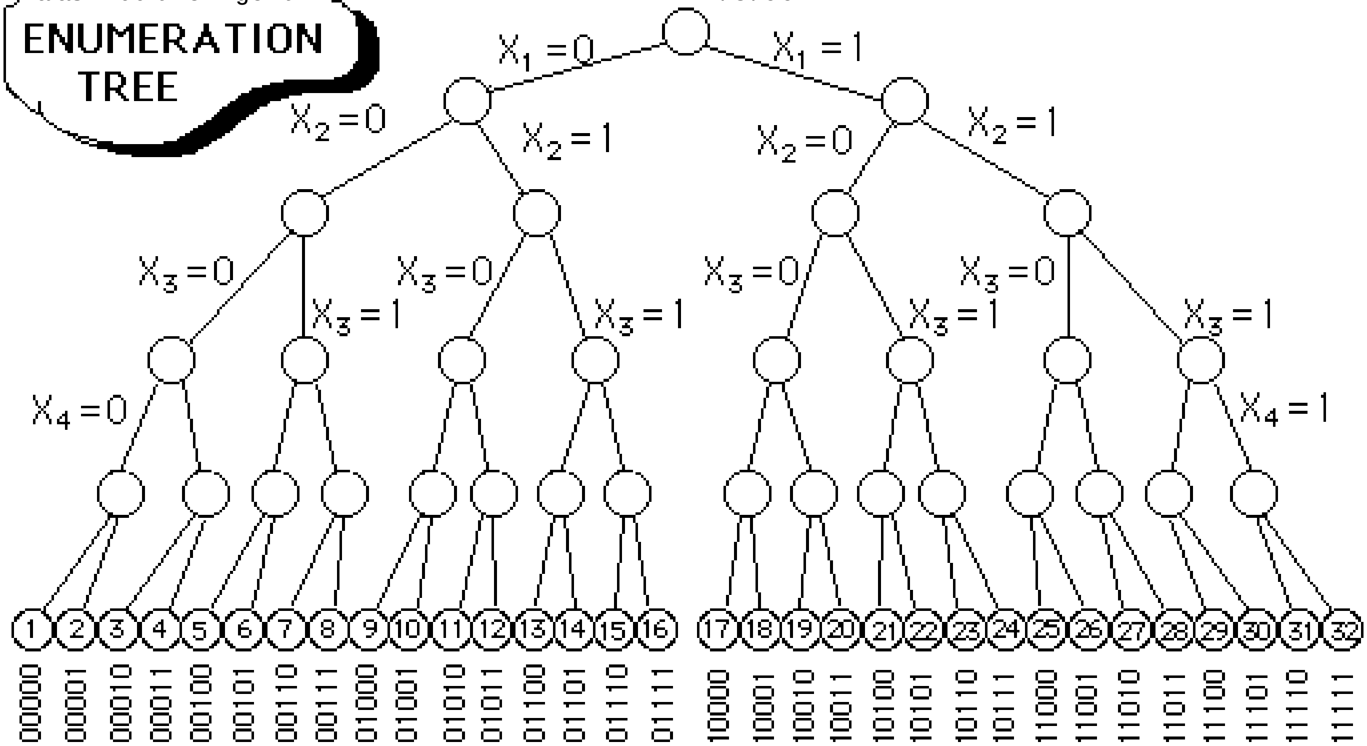
$$\begin{cases} z = 3X_1 + 8X_2 + X_3 + 16X_4 + X_5 \\ g_1(X) = X_1 - 2X_2 - 6X_3 + 2X_4 + 3X_5 \le 0 \\ g_2(X) = X_1 - 3X_3 - 2X_4 + 2X_5 \le -2 \\ g_3(X) = X_1 - 5X_2 + 4X_3 - X_4 - 2X_5 \le -5 \end{cases}$$

| # | X | z | $g_1$ | $g_2$ | $g_3$ |
|---|---|---|---|---|---|
| 1 | 0 0 0 0 0 | 0 | 0 | 0 | 0 |
| 2 | 0 0 0 0 1 | 1 | 3 | 2 | -2 |
| 3 | 0 0 0 1 0 | 16 | 2 | -2 | -1 |
| 4 | 0 0 0 1 1 | 17 | 5 | 0 | -3 |
| 5 | 0 0 1 0 0 | 1 | -6 | -3 | 4 |
| 6 | 0 0 1 0 1 | 2 | -3 | -1 | 2 |
| 7 | 0 0 1 1 0 | 17 | -4 | -5 | 3 |
| 8 | 0 0 1 1 1 | 18 | -1 | -3 | 1 |
| 9 | 0 1 0 0 0 | 8 | -2 | 0 | -5 |
| 10 | 0 1 0 0 1 | 9 | 1 | 2 | -7 |
| 11 | 0 1 0 1 0 | 24 | 0 | -2 | -6 |
| 12 | 0 1 0 1 1 | 25 | 3 | 0 | -8 |
| 13 | 0 1 1 0 0 | 9 | -8 | -3 | -1 |
| 14 | 0 1 1 0 1 | 10 | -5 | -1 | -3 |
| 15 | 0 1 1 1 0 | 25 | -6 | -5 | -2 |
| 16 | 0 1 1 1 1 | 26 | -3 | -3 | -4 |
| 17 | 1 0 0 0 0 | 3 | 1 | 1 | 1 |
| 18 | 1 0 0 0 1 | 4 | 4 | 3 | -1 |
| 19 | 1 0 0 1 0 | 19 | 3 | -1 | 0 |
| 20 | 1 0 0 1 1 | 20 | 6 | 1 | -2 |
| 21 | 1 0 1 0 0 | 4 | -5 | -2 | 5 |
| 22 | 1 0 1 0 1 | 5 | -2 | 0 | 3 |
| 23 | 1 0 1 1 0 | 20 | -3 | -4 | 4 |
| 24 | 1 0 1 1 1 | 21 | 0 | -2 | 2 |
| 25 | 1 1 0 0 0 | 11 | -1 | 1 | -4 |
| 26 | 1 1 0 0 1 | 12 | 2 | 3 | -6 |
| 27 | 1 1 0 1 0 | 27 | 1 | -1 | -5 |
| 28 | 1 1 0 1 1 | 28 | 4 | 1 | -7 |
| 29 | 1 1 1 0 0 | 12 | -7 | -2 | 0 |
| 30 | 1 1 1 0 1 | 13 | -4 | 0 | -2 |
| 31 | 1 1 1 1 0 | 25 | -5 | -4 | -1 |
| 32 | 1 1 1 1 1 | 26 | -2 | -2 | -3 |

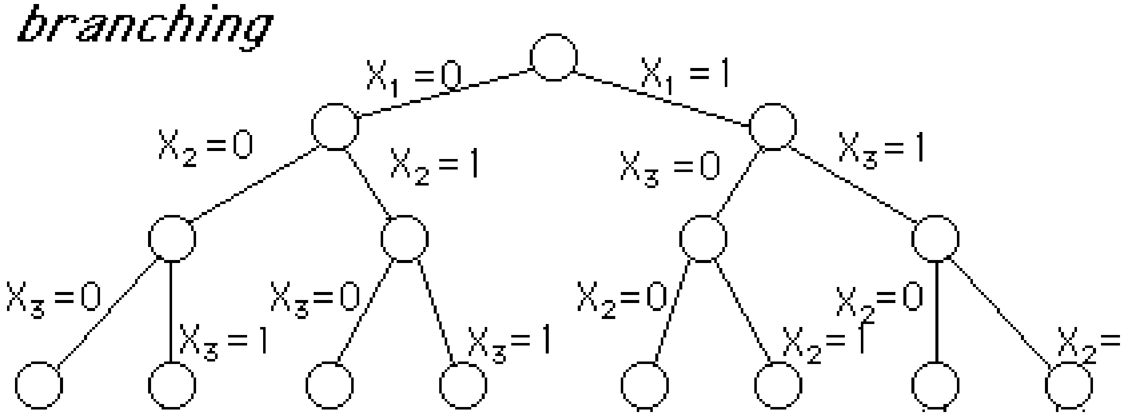

| # | X | z | $g_1$ | $g_2$ | $g_3$ |
|---|---|---|---|---|---|
| 1 | 0 0 0 0 0 | 0 | 0 | 0 | 0 |
| 2 | 0 0 0 0 1 | 1 | 3 | 2 | -2 |
| 3 | 0 0 0 1 0 | 16 | 2 | -2 | -1 |
| 4 | 0 0 0 1 1 | 17 | 5 | 0 | -3 |
| 5 | 0 0 1 0 0 | 1 | -6 | -3 | 4 |
| 6 | 0 0 1 0 1 | 2 | -3 | -1 | 2 |
| 7 | 0 0 1 1 0 | 17 | -4 | -5 | 3 |
| 8 | 0 0 1 1 1 | 18 | -1 | -3 | 1 |
| 9 | 0 1 0 0 0 | 8 | -2 | 0 | -5 |
| 10 | 0 1 0 0 1 | 9 | 1 | 2 | -7 |
| 11 | 0 1 0 1 0 | 24 | 0 | -2 | -6 |
| 12 | 0 1 0 1 1 | 25 | 3 | 0 | -8 |
| 13 | 0 1 1 0 0 | 9 | -8 | -3 | -1 |
| 14 | 0 1 1 0 1 | 10 | -5 | -1 | -3 |
| 15 | 0 1 1 1 0 | 25 | -6 | -5 | -2 |
| 16 | 0 1 1 1 1 | 26 | -3 | -3 | -4 |
| 17 | 1 0 0 0 0 | 3 | 1 | 1 | 1 |
| 18 | 1 0 0 0 1 | 4 | 4 | 3 | -1 |
| 19 | 1 0 0 1 0 | 19 | 3 | -1 | 0 |
| 20 | 1 0 0 1 1 | 20 | 6 | 1 | -2 |
| 21 | 1 0 1 0 0 | 4 | -5 | -2 | 5 |
| 22 | 1 0 1 0 1 | 5 | -2 | 0 | 3 |
| 23 | 1 0 1 1 0 | 20 | -3 | -4 | 4 |
| 24 | 1 0 1 1 1 | 21 | 0 | -2 | 2 |
| 25 | 1 1 0 0 0 | 11 | -1 | 1 | -4 |
| 26 | 1 1 0 0 1 | 12 | 2 | 3 | -6 |
| 27 | 1 1 0 1 0 | 27 | 1 | -1 | -5 |
| 28 | 1 1 0 1 1 | 28 | 4 | 1 | -7 |
| 29 | 1 1 1 0 0 | 12 | -7 | -2 | 0 |
| 30 | 1 1 1 0 1 | 13 | -4 | 0 | -2 |
| 31 | 1 1 1 1 0 | 25 | -5 | -4 | -1 |
| 32 | 1 1 1 1 1 | 26 | -2 | -2 | -3 |


Solution #11 is the only one feasible in all 3 constraints

# ENUMERATION TREE



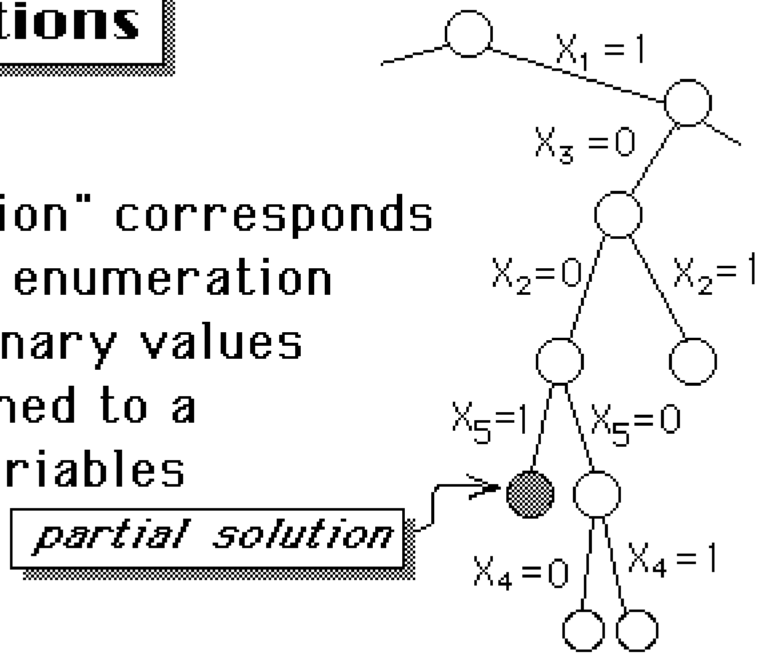

*The order of branching is not important, e.g., one can branch on $X_3$ before branching on $X_2$*



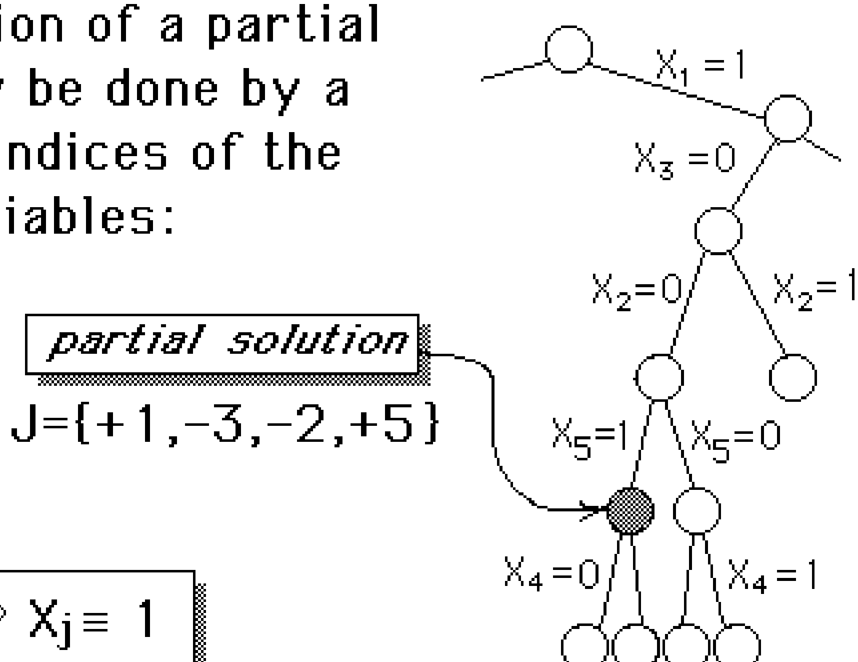

*In fact, the choice of branching variable may differ on the same level of the tree!*

# Partial Solutions

A "partial solution" corresponds to a node of the enumeration tree in which binary values have been assigned to a subset of the variables



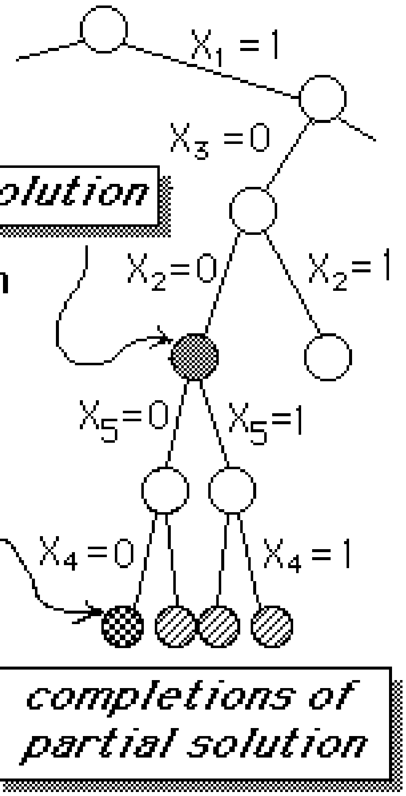

Representation of a partial solution may be done by a vector of ± indices of the assigned variables:




$\{\ldots,+j,\ldots\} \Rightarrow X_j \equiv 1$

$\{\ldots,-j,\ldots\} \Rightarrow X_j \equiv 0$

# Completions




*partial solution*

The completions of a partial solution consist of ALL of the nodes at the bottom-most level of the tree, where all variables have been assigned.

zero completion

The completion with all free variables assigned value of zero is the "zero completion"

# Fathoming of a Partial Solution

A partial solution (node) of an enumeration tree may be considered fathomed if one of the following may be demonstrated:

- all completions violate one or more constraints
- all completions are inferior (with respect to the objective) to the incumbent
- the zero completion is feasible & superior to the incumbent (& therefore becomes the new incumbent)

# Fathoming Test #1

A free variable $X_j$ $(j \notin J)$ which has nonnegative coefficients in *every* constraint which is violated by the zero completion should be zero, since assigning it the value 1 will improve neither the objective function nor feasibility.

Compute

$$A = \left\{ j \mid j \in N - J, a_{ij} \geq 0 \ \forall \ i \in M \ \text{ such that } S_i < 0 \right\}$$

and

$$N^1 = N - J - A$$

*indices of free variables which are eligible to be assigned value 1*

If $N^1 = \varnothing$, then the partial solution J may be fathomed!

FATHOMING TEST ONE

# Fathoming Test #2

Let $Z$ be the objective function value of the zero completion of the partial solution J.

If $Z + C_k \geq \underline{Z}$ (the incumbent) for some $k \notin J$, then no completion of J which has $X_k = 1$ can be optimal!

Compute $$B = \left\{ j \mid j \in N^1, Z + C_j \geq \underline{Z} \right\}$$

and $$N^2 = N^1 - B$$

*indices of all free variables which are eligible to be assigned value 1*

If $N^2 = \varnothing$, then the partial solution may be fathomed!

**FATHOMING TEST TWO**

# Fathoming Test #3

If constraint #i is violated by the zero completion of the partial solution, so that the slack $S_i < 0$,

and if the sum of all negative coefficients of the free variables (in $N^2$ ) exceeds $S_i$,

Then no feasible completion of the partial solution exists.

Compute

$$C = \left\{ i \;\middle|\; S_i < \sum_{j \in N^2} a_{ij}^- \right\}$$

If $C \neq \varnothing$ then the partial solution is fathomed.

FATHOMING TEST THREE

# Selection of a Free Variable for Forward Step

*When the fathoming tests fail to fathom the current partial solution, branching will be performed, by fixing a free variable $X_j$*

$$J \leftarrow J, \{+j\}$$

*The positive index "j" is appended to the end of the current J vector*

*Any free variable might be chosen....*
*is there a "best" choice?*

Let $S_i$ = slack in constraint #i in the zero completion of J

Then $S_i - a_{ij}$ = slack in constraint #i if free variable $X_j=1$ while other free variables are assigned value zero

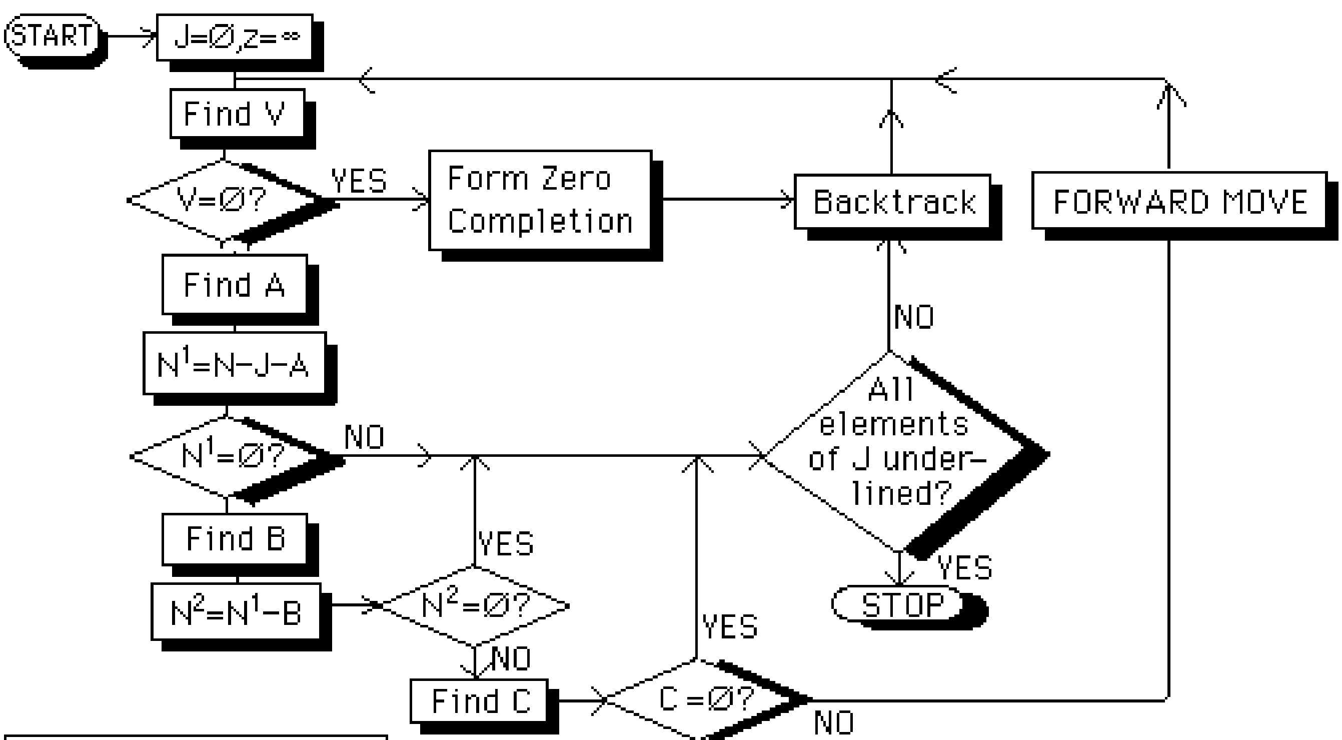
Define $(S_i - a_{ij})^- = \min\{0, S_i - a_{ij}\}$ *NEGATIVE PART*

$v_j = \sum_i (S_i - a_{ij})^-$ measures the infeasibility which results from fixing $X_j=1$

Balas' strategy was to choose the free variable which would result in the *least* infeasibility, i.e., the maximum ("least negative") value of $v_j$

$$j^* = \underset{j \in N^2}{\operatorname{argmax}} \{v_j\} = \underset{j \in N^2}{\operatorname{argmax}} \sum_i (s_i - a_{ij})^-$$

*Other rules might result in partial solutions which are more easily fathomed.*




**Flowchart**

$$\text{Minimize} \quad 4X_1 + 8X_2 + 9X_3 + 3X_4 + 4X_5 + 10X_6$$

$$\text{s.t.} \quad \begin{cases} 4X_1 - 5X_2 - 3X_3 - 2X_4 - X_5 + 8X_6 \le -8 \\ -5X_1 + 2X_2 + 9X_3 + 8X_4 - 3X_5 + 8X_6 \le 7 \\ 8X_1 + 5X_2 - 4X_3 + X_5 + 6X_6 \le 6 \end{cases}$$

$$X_j \in \{0, 1\} \ \forall \ j=1, \ldots, 6$$

# Inserting slack variables:

$$4X_1 - 5X_2 - 3X_3 - 2X_4 - X_5 + 8X_6 + S_1 = -8$$
$$-5X_1 + 2X_2 + 9X_3 + 8X_4 - 3X_5 + 8X_6 + S_2 = 7$$
$$8X_1 + 5X_2 - 4X_3 + X_5 + 6X_6 + S_3 = 6$$

Random ILP (seed = 148458)

\# variables = 6
\# constraints = 3

| 1 | 2 | 3 | 4 | 5 | 6 | | b |
|---|---|---|---|---|---|---|---|
| 4 | 8 | 9 | 3 | 4 | 10 | | min |
| 4 | -5 | -3 | -2 | -1 | 8 | ≤ | -8 |
| -5 | 2 | 9 | 8 | -3 | 8 | ≤ | 7 |
| 8 | 5 | -4 | 0 | 1 | 6 | ≤ | 6 |

Constraints are of the form $Ax \le b$

| | J | V1 | A | N1 | B | N2 | C | v | j | Z* |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | 1 | 1 6 | 2 3 4 5 | | 2 3 4 5 | | -3 -7 -7 -7 | 2 | *** |

(1) $J = \varnothing$

Constraints violated by zero completion:

$S_1 = -8$ ← violation!
$S_2 = 7$ ok
$S_3 = 6$ ok

A = {1,6}: variables which cannot improve feasibility in violated constraints if equal to 1

$$4\,X_1 - 5\,X_2 - 3\,X_3 - 2\,X_4 - X_5 + 8\,X_6 + S_1 = -8$$

Constraint #1

*nonnegative coefficients in violated constraint!*

| | J | V1 | A | N1 | B | N2 | C | v | j | Z* |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | 1 | 1 6 | 2 3 4 5 | | 2 3 4 5 | | -3 -7 -7 -7 | 2 | *** |

(1) $J = \varnothing$

$N^1 = N - J - A = \{1,2,3,4,5,6\} - \varnothing - \{1,6\} = \{2,3,4,5\}$

Indices of free variables which might be assigned value of 1

**FATHOMING TEST #1**

$N^1 \neq \varnothing$ , so this test fails to fathom the partial solution!

| | J | V1 | A | N1 | B | N2 | C | v | j | Z* |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | 1 | 1 6 | 2 3 4 5 | | 2 3 4 5 | | -3 -7 -7 -7 | 2 | *** |

(1) $J = \emptyset$ Fathoming Test #2 isn't applicable, since we do not yet have a finite incumbent.

$$4X_1 - 5X_2 - 3X_3 - 2X_4 - X_5 + 8X_6 + S_1 = -8$$

It is possible to satisfy constraint #1 by assigning values to the free variables having negative coefficients, e.g.,

$$X_2=X_3=X_4=X_5=1 \Rightarrow S_1 = -8 +5 +3 +2 +1 = 3 > 0$$

feasible! $\Rightarrow C = \emptyset$

**FATHOMING TEST #3**

This test fails to fathom the partial sol'n

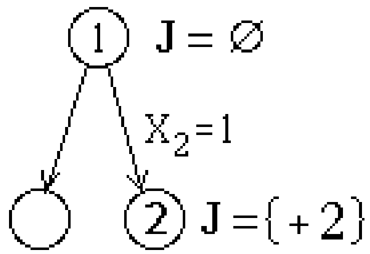
| | J | V1 | A | N1 | B | N2 | C | v | j | Z* |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | 1 | 1 6 | 2 3 4 5 | | 2 3 4 5 | | -3 -7 -7 -7 | 2 | *** |

(1) $J = \emptyset$ Since the fathoming tests have all failed, we must next choose a variable for branching.

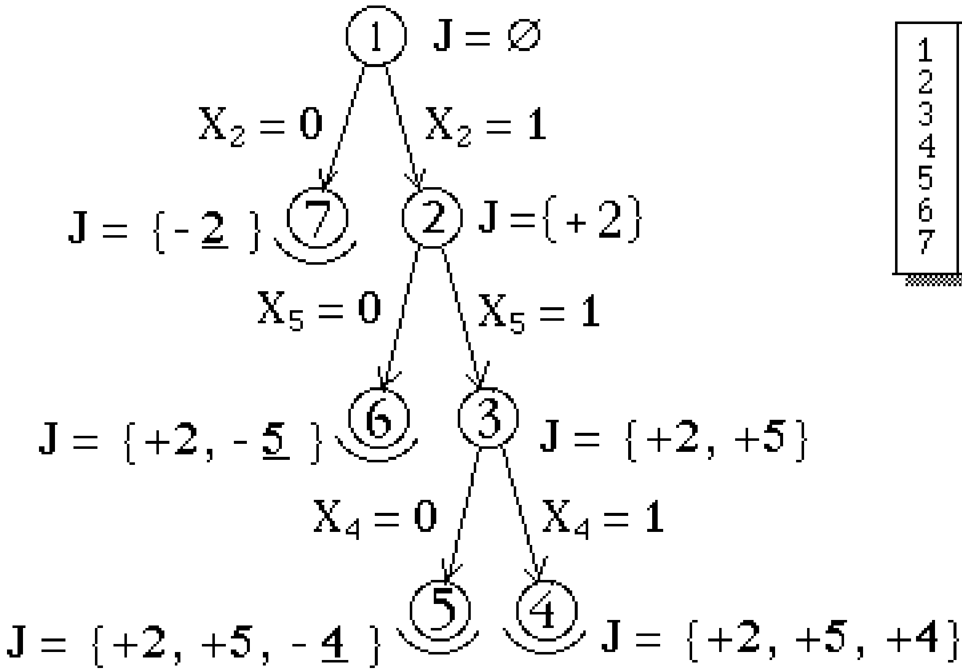
| Variable | constraint infeasibility if =1 | | | Total |
|---|---|---|---|---|
| | 1 | 2 | 3 | |
| 2 | -3 | 0 | 0 | -3 |
| 3 | -5 | -2 | 0 | -7 |
| 4 | -6 | -1 | 0 | -7 |
| 5 | -7 | 0 | 0 | -7 |

Least amount of infeasibility if assigned 1

| | J | V1 | A | N1 | B | N2 | C | v | j | Z* |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | 1 | 1 6 | 2 3 4 5 | | 2 3 4 5 | | −3 −7 −7 −7 | 2 | *** |
| 2 | 2 | 1 | 1 6 | 3 4 5 | | 3 4 5 | | −4 −4 −2 | 5 | *** |




| | J | V1 | A | N1 | B | N2 | C | v | j | Z* |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | 1 | 1 6 | 2 3 4 5 | | 2 3 4 5 | | −3 −7 −7 −7 | 2 | *** |
| 2 | 2 | 1 | 1 6 | 3 4 5 | | 3 4 5 | | −4 −4 −2 | 5 | *** |
| 3 | 2 5 | 1 | 1 6 | 3 4 | | 3 4 | | −1 | 4 | *** |
| 4 | 2 5 4 | | | | | | | | | *** |
| 5 | 2 5 −4 | 1 | 1 6 | 3 | 3 | | | | | 15 |
| 6 | 2 −5 | 1 | 1 6 | 3 4 | 3 | 4 | 1 | | | 15 |
| 7 | −2 | 1 | 1 6 | 3 4 5 | | 3 4 5 | 1 | | | 15 |




| | J | | |
|---|---|---|---|
| 1 | | | |
| 2 | 2 | | |
| 3 | 2 | 5 | |
| 4 | 2 | 5 | 4 |
| 5 | 2 | 5 | -4 |
| 6 | 2 | -5 | |
| 7 | -2 | | |

Random ILP (seed = 148458)

Solution is:

| i | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| X[i] | 0 | 1 | 0 | 1 | 1 | 0 |

Objective function value is 15

## Example Problem

\# variables = 5
\# constraints = 3

| 1 | 2 | 3 | 4 | 5 | | b |
|---|---|---|---|---|---|---|
| 5 | 7 | 10 | 3 | 1 | | min |
| ¯1 | 3 | ¯5 | ¯1 | 4 | ≤ | ¯2 |
| 2 | ¯6 | 3 | 2 | ¯2 | ≤ | 0 |
| 0 | 1 | ¯2 | 1 | 1 | ≤ | ¯1 |

Constraints are of the form Ax≤b

| iteration | J | V1 | A | N1 | B | N2 | C | v | j | Z* |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | 1 3 | 2 5 | 1 3 4 | *** | 1 3 4 | | ¯4 ¯3 ¯5 | 3 | *** |
| 2 | 3 | 2 | 1 4 | 2 5 | *** | 2 5 | | ¯2 | 2 | *** |
| 3 | 3 2 | | | | *** | | | | | *** |
| 4 | 3 ¯2 | 2 | 1 4 | 5 | *** | 5 | 2 | | | 17 |
| 5 | ¯3 | 1 3 | 2 5 | 1 4 | *** | 1 4 | 3 | | | 17 |

Balas' Additive Algorithm

## Example Problem

CPU time= 1.75 sec.

Solution is:

| i | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| X[i] | 0 | 1 | 1 | 0 | 0 |

Objective function value is 17

## Random ILP (seed = 825025)

\# variables = 8
\# constraints = 5

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | b |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 4 | 5 | 9 | 5 | 9 | 4 | 6 | | min |
| 3 | ¯5 | 4 | ¯2 | 6 | ¯4 | 6 | ¯5 | ≤ | ¯2 |
| 0 | 8 | 1 | 8 | ¯2 | 2 | 0 | 4 | ≤ | 7 |
| 9 | 2 | 4 | 7 | ¯3 | 2 | 6 | 1 | ≤ | 16 |
| ¯5 | 2 | 5 | ¯2 | 6 | ¯4 | 0 | 4 | ≤ | 0 |
| 9 | ¯1 | 1 | 1 | ¯3 | 6 | 7 | 0 | ≤ | 16 |

Constraints are of the form Ax≤b

| | J | V1 | A |
|---|---|---|---|
| 1 | | 1 | 1 3 5 7 |
| 2 | 6 | | |
| 3 | ¯6 | 1 | 1 3 5 7 |
| 4 | ¯6 2 | 2 4 | 3 7 8 |
| 5 | ¯6 ¯2 | 1 | 1 3 5 7 |
| 6 | ¯6 ¯2 8 | 4 | 3 5 7 |
| 7 | ¯6 ¯2 ¯8 | 1 | 1 3 5 7 |

| N1 | B | N2 | C | v | j | Z* |
|---|---|---|---|---|---|---|
| 2 4 6 8 | | 2 4 6 8 | | ¯3 ¯1 0 ¯4 | 6 | *** |
| | | | | | | *** |
| 2 4 8 | 4 | 2 8 | | ¯3 ¯4 | 2 | 9 |
| 1 4 5 | 4 5 | 1 | 2 | | | 9 |
| 4 8 | 4 | 8 | | ¯4 | 8 | 9 |
| 1 4 | 1 4 | | | | | 9 |
| 4 | 4 | | | | | 9 |

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | b |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 4 | 5 | 9 | 5 | 9 | 4 | 6 | | min |
| 3 | ¯5 | 4 | ¯2 | 6 | ¯4 | 6 | ¯5 | ≤ | ¯2 |
| 0 | 8 | 1 | 8 | ¯2 | 2 | 0 | 4 | ≤ | 7 |
| 9 | 2 | 4 | 7 | ¯3 | 2 | 6 | 1 | ≤ | 16 |
| ¯5 | 2 | 5 | ¯2 | 6 | ¯4 | 0 | 4 | ≤ | 0 |
| 9 | ¯1 | 1 | 1 | ¯3 | 6 | 7 | 0 | ≤ | 16 |

| | J | V1 | A | N1 | B | N2 | C | v | j | Z* |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | 1 | 1 3 5 7 | 2 4 6 8 | | 2 4 6 8 | | ¯3 ¯1 0 ¯4 | 6 | *** |

The first constraint is violated by the zero completion (S =-2).

Variables 1,3,5, &7 have positive coefficients in this constraint, and thus cannot help in achieving feasibility. They form the set A, which are implicitly fixed = 0, leaving N = {2, 4, 6, 8}.

Test 2 isn't applicable because no incumbent has been identified.

Test 3 considers the violated constraints in V1 to determine whether it is possible to satisfy them. In this case, we see that increasing any one of variables 2,4,6, or 8 will result in feasibility, so C is empty.

The fathoming tests have failed, and therefore we must perform a forward branch.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | b |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 4 | 5 | 9 | 5 | 9 | 4 | 6 | | min |
| 3 | -5 | 4 | -2 | 6 | -4 | 6 | -5 | ≤ | -2 |
| 0 | 8 | 1 | 8 | -2 | 2 | 0 | 4 | ≤ | 7 |
| 9 | 2 | 4 | 7 | -3 | 2 | 6 | 1 | ≤ | 16 |
| -5 | 2 | 5 | -2 | 6 | -4 | 0 | 4 | ≤ | 0 |
| 9 | -1 | 1 | 1 | -3 | 6 | 7 | 0 | ≤ | 16 |

| | J | V1 | A | N1 | B | N2 | C | v | j | Z* |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | 1 | 1 3 5 7 | 2 4 6 8 | | 2 4 6 8 | | -3 -1 0 -4 | 6 | *** |

Choosing the branching variable:

Setting variable 2 equal to 1 results in constraint violations {0, 1, 0, 2, 0} and so V2 = -3.

Setting variable 4 equal to 1 results in constraint violations {0, 1, 0, 0, 0} and so V4 = -1

Setting variable 6 equal to 1 results in constraint violations {0, 0, 0, 0, 0} and so V6=0.

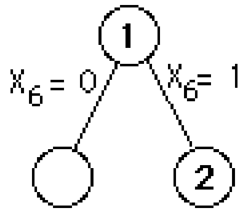
Setting variable 8 equal to 1 results in constraint violations {0, 0, 0, 4, 0} and so V8 = 0.


| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | b |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 4 | 5 | 9 | 5 | 9 | 4 | 6 | | min |
| 3 | -5 | 4 | -2 | 6 | -4 | 6 | -5 | ≤ | -2 |
| 0 | 8 | 1 | 8 | -2 | 2 | 0 | 4 | ≤ | 7 |
| 9 | 2 | 4 | 7 | -3 | 2 | 6 | 1 | ≤ | 16 |
| -5 | 2 | 5 | -2 | 6 | -4 | 0 | 4 | ≤ | 0 |
| 9 | -1 | 1 | 1 | -3 | 6 | 7 | 0 | ≤ | 16 |

| | J | V1 | A | N1 | B | N2 | C | v | j | Z* |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | 1 | 1 3 5 7 | 2 4 6 8 | | 2 4 6 8 | | -3 -1 0 -4 | 6 | *** |


The (rather arbitrary) rule is to select that variable causing the least infeasibility, and so variable 6 is selected for the branching.

Therefore, J, which was previously empty, is now {+6}.

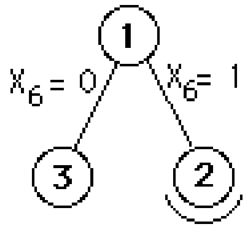



| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | b |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 4 | 5 | 9 | 5 | 9 | 4 | 6 | | min |
| 3 | -5 | 4 | -2 | 6 | -4 | 6 | -5 | ≤ | -2 |
| 0 | 8 | 1 | 8 | -2 | 2 | 0 | 4 | ≤ | 7 |
| 9 | 2 | 4 | 7 | -3 | 2 | 6 | 1 | ≤ | 16 |
| -5 | 2 | 5 | -2 | 6 | -4 | 0 | 4 | ≤ | 0 |
| 9 | -1 | 1 | 1 | -3 | 6 | 7 | 0 | ≤ | 16 |

| | J | V1 | A | N1 | B | N2 | C | v | j | Z* |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 6 | | | | | | | | | *** |

At node 2, J={+6} and no constraints are violated by the zero completion (i.e., X = 1 and all other variables zero).
Since no other completion of this partial solution can cost less than the zero completion, the node is fathomed, and we may backtrack.

Backtracking: J becomes {-6}




| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | b |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 4 | 5 | 9 | 5 | 9 | 4 | 6 | | min |
| 3 | ⁻5 | 4 | ⁻2 | 6 | ⁻4 | 6 | ⁻5 | ≤ | ⁻2 |
| 0 | 8 | 1 | 8 | ⁻2 | 2 | 0 | 4 | ≤ | 7 |
| 9 | 2 | 4 | 7 | ⁻3 | 2 | 6 | 1 | ≤ | 16 |
| ⁻5 | 2 | 5 | ⁻2 | 6 | ⁻4 | 0 | 4 | ≤ | 0 |
| 9 | ⁻1 | 1 | 1 | ⁻3 | 6 | 7 | 0 | ≤ | 16 |

| | J | V1 | A | N1 | B | N2 | C | v | j | Z* |
|---|---|---|---|---|---|---|---|---|---|---|
| 3 | ⁻6 | 1 | 1 3 5 7 | 2 4 8 | 4 | 2 8 | | ⁻3 ⁻4 | 2 | 9 |

At node 3, again only the first constraint is violated by the zero completion, and variables 1, 3, 5, & 7 cannot contribute toward making this constraint feasible, so that they are implicitly fixed at value zero, leaving only free variables 2, 4, & 8.

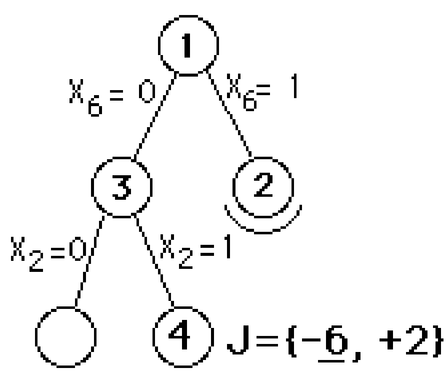
If X2 or X8 were fixed at value 1, the objective function is less than the incumbent, but if X4 were fixed at 1, the objective function would exceed the incumbent (B = {4}) and therefore is implicitly fixed at value 0, leaving only N = {2, 8} as free variables. Fixing either of these at value 1 would satisfy the violated constraint (#1), so C is empty.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | b |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 4 | 5 | 9 | 5 | 9 | 4 | 6 | | min |
| 3 | -5 | 4 | -2 | 6 | -4 | 6 | -5 | ≤ | -2 |
| 0 | 8 | 1 | 8 | -2 | 2 | 0 | 4 | ≤ | 7 |
| 9 | 2 | 4 | 7 | -3 | 2 | 6 | 1 | ≤ | 16 |
| -5 | 2 | 5 | -2 | 6 | -4 | 0 | 4 | ≤ | 0 |
| 9 | -1 | 1 | 1 | -3 | 6 | 7 | 0 | ≤ | 16 |

| | J | V1 | A | N1 | B | N2 | C | v | j | Z* |
|---|---|---|---|---|---|---|---|---|---|---|
| 3 | -6 | 1 | 1 3 5 7 | 2 4 8 | 4 | 2 8 | | -3 -4 | 2 | 9 |

Therefore we cannot fathom this node, and must make a forward move, i.e., branch.

Selection of branching variable: Fixing variable 2 at 1 gives constraint violations 0, 0, 1, 0, 2, 0, while fixing variable 8 at 1 gives violations 0, 0, 0, 4, 0. Variable 2 results in less infeasibility, and is selected for branching.



| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | b |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 4 | 5 | 9 | 5 | 9 | 4 | 6 | | min |
| 3 | −5 | 4 | −2 | 6 | −4 | 6 | −5 | ≤ | −2 |
| 0 | 8 | 1 | 8 | −2 | 2 | 0 | 4 | ≤ | 7 |
| 9 | 2 | 4 | 7 | −3 | 2 | 6 | 1 | ≤ | 16 |
| −5 | 2 | 5 | −2 | 6 | −4 | 0 | 4 | ≤ | 0 |
| 9 | −1 | 1 | 1 | −3 | 6 | 7 | 0 | ≤ | 16 |

| | J | V1 | A | N1 | B | N2 | C | v | j | Z* |
|---|---|---|---|---|---|---|---|---|---|---|
| 4 | −6 2 | 2 4 | 3 7 8 | 1 4 5 | 4 5 | 1 | 2 | | | 9 |

At node 4, constraints 2 & 4 are violated by the zero completion, but variables 3, 7, & 8 cannot assist in making these constraints feasible, and are therefore implicitly set equal to zero, leaving variables 1, 4, &5 as free variables.

Consider X4: together with X2 this gives a cost of 13, exceeding the incumbent (9); likewise, variable X5 together with X2 gives a cost of 9 which is no better than the incumbent. Hence variables 4&5 may be implicitly fixed at value zero, leaving only variable 1 as a free variable.

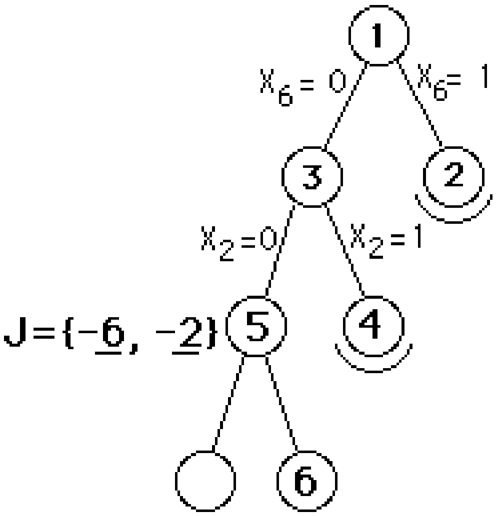

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | b |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 4 | 5 | 9 | 5 | 9 | 4 | 6 | | min |
| 3 | −5 | 4 | −2 | 6 | −4 | 6 | −5 | ≤ | −2 |
| 0 | 8 | 1 | 8 | −2 | 2 | 0 | 4 | ≤ | 7 |
| 9 | 2 | 4 | 7 | −3 | 2 | 6 | 1 | ≤ | 16 |
| −5 | 2 | 5 | −2 | 6 | −4 | 0 | 4 | ≤ | 0 |
| 9 | −1 | 1 | 1 | −3 | 6 | 7 | 0 | ≤ | 16 |

| | J | V1 | A | N1 | B | N2 | C | v | j | Z* |
|---|---|---|---|---|---|---|---|---|---|---|
| 4 | −6 2 | 2 4 | 3 7 8 | 1 4 5 | 4 5 | 1 | 2 | | | 9 |


With variable 2 equal to 1 and only variable 1 free, we can determine that the violated constraint #2 cannot be made feasible. (Constraint 4 could be made feasible by setting X1 = 1.) Hence C={2} and the subproblem is fathomed.

We must now backtrack:

Currently J = {−6, +2} and so the next node will have J={+6, −2}.




| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | b |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 4 | 5 | 9 | 5 | 9 | 4 | 6 | | min |
| 3 | -5 | 4 | -2 | 6 | -4 | 6 | -5 | ≤ | -2 |
| 0 | 8 | 1 | 8 | -2 | 2 | 0 | 4 | ≤ | 7 |
| 9 | 2 | 4 | 7 | -3 | 2 | 6 | 1 | ≤ | 16 |
| -5 | 2 | 5 | -2 | 6 | -4 | 0 | 4 | ≤ | 0 |
| 9 | -1 | 1 | 1 | -3 | 6 | 7 | 0 | ≤ | 16 |

| | J | V1 | A | N1 | B | N2 | C | v | j | Z* |
|---|---|---|---|---|---|---|---|---|---|---|
| 5 | -6 -2 | 1 | 1 3 5 7 | 4 8 | 4 | 8 | | -4 | 8 | 9 |

At node 5, variables 2 & 6 are zero, and again constraint 1 is violated by the zero completion.

Variables 1, 3, 5, & 7 cannot help to achieve feasibility of this constraint (since they have positive coefficients) and therefore they can be made implicitly zero, leaving only variables 4 & 8 as free variables.

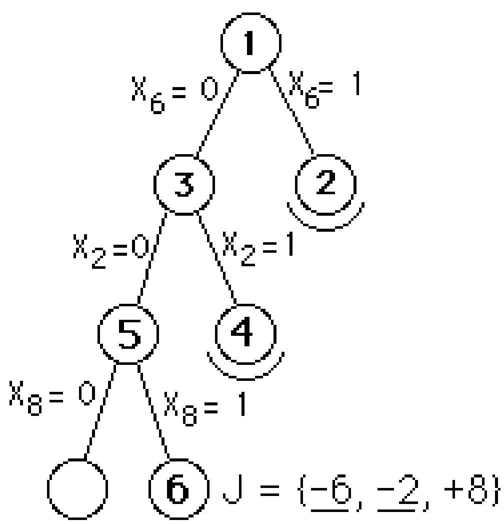
Variable 4, if set = 1, would cause the cost to exceed the incumbent, and therefore is implicitly fixed at zero, leaving only variable 8 free

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | b |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 4 | 5 | 9 | 5 | 9 | 4 | 6 | | min |
| 3 | −5 | 4 | −2 | 6 | −4 | 6 | −5 | ≤ | −2 |
| 0 | 8 | 1 | 8 | −2 | 2 | 0 | 4 | ≤ | 7 |
| 9 | 2 | 4 | 7 | −3 | 2 | 6 | 1 | ≤ | 16 |
| −5 | 2 | 5 | −2 | 6 | −4 | 0 | 4 | ≤ | 0 |
| 9 | −1 | 1 | 1 | −3 | 6 | 7 | 0 | ≤ | 16 |

| | J | V1 | A | N1 | B | N2 | C | v | j | Z* |
|---|---|---|---|---|---|---|---|---|---|---|
| 5 | −6 −2 | 1 | 1 3 5 7 | 4 8 | 4 | 8 | | −4 | 8 | 9 |

We see that with only variable 8, it is possible to satisfy constraint 1 (by setting X8 = 1), so C is empty.
Fixing X8=1 results in infeasibilities 0, 0, 0, 4, 0. Obviously variable 8 is chosen for the branching.

J, which was {−6, −2}, is extended on the right by +8, i.e.,
J = {−6, −2, +8}.

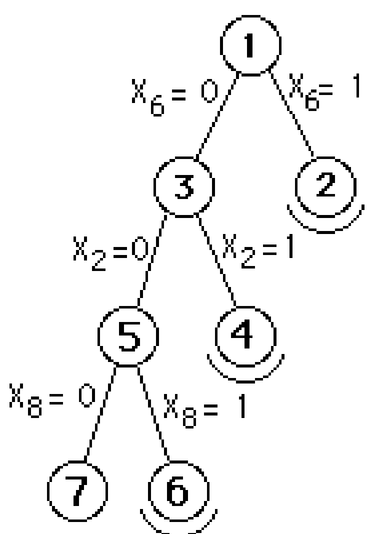


| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | b |
|---|---|---|---|---|---|---|---|---|---|---|
| | 3 | 4 | 5 | 9 | 5 | 9 | 4 | 6 | | min |
| | 3 | -5 | 4 | -2 | 6 | -4 | 6 | -5 | ≤ | -2 |
| | 0 | 8 | 1 | 8 | -2 | 2 | 0 | 4 | ≤ | 7 |
| | 9 | 2 | 4 | 7 | -3 | 2 | 6 | 1 | ≤ | 16 |
| | -5 | 2 | 5 | -2 | 6 | -4 | 0 | 4 | ≤ | 0 |
| | 9 | -1 | 1 | 1 | -3 | 6 | 7 | 0 | ≤ | 16 |

| | J | V1 | A | N1 | B | N2 | C | v | j | Z* |
|---|---|---|---|---|---|---|---|---|---|---|
| 6 | -6 -2 8 | 4 | 3 5 7 | 1 4 | 1 4 | | | | | 9 |

At node 6, the zero completion violates constraint 4, and the free variables 3, 5, & 7 cannot help to remove the feasibility, and hence are implicity fixed at value zero, leaving only variables 1 & 4 as free variables.

However, increasing variable 1 would result in a cost of 6+3, which is no better than the incumbent, while increasing variable 4 would result in a cost of 15, worse than the incumbent. These two variables are implicitly fixed at value zero, therefore, leaving no free variables. The node is fathomed.



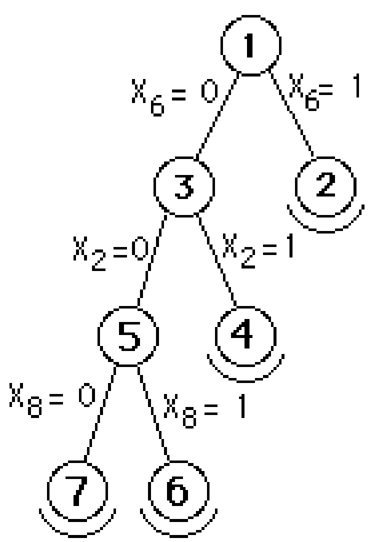

J = {-6, -2, -8}

To backtrack from J={-6, -2, +8}, we look for the last element without underline, reverse its sign, and underline it, giving us

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | b |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 4 | 5 | 9 | 5 | 9 | 4 | 6 | | min |
| 3 | ⁻5 | 4 | ⁻2 | 6 | ⁻4 | 6 | ⁻5 | ≤ | ⁻2 |
| 0 | 8 | 1 | 8 | ⁻2 | 2 | 0 | 4 | ≤ | 7 |
| 9 | 2 | 4 | 7 | ⁻3 | 2 | 6 | 1 | ≤ | 16 |
| ⁻5 | 2 | 5 | ⁻2 | 6 | ⁻4 | 0 | 4 | ≤ | 0 |
| 9 | ⁻1 | 1 | 1 | ⁻3 | 6 | 7 | 0 | ≤ | 16 |

| | J | V1 | A | N1 | B | N2 | C | v | j | Z* |
|---|---|---|---|---|---|---|---|---|---|---|
| 7 | ⁻6 ⁻2 ⁻8 | 1 | 1 3 5 7 | 4 | 4 | | | | | 9 |

At node 7, variables 2, 6, &8 are all fixed at zero, and the first constraint is violated by the zero completion. Variables 1, 3, 5, and 7 all have positive coefficients in this constraint and are therefore unable to assist in gaining feasibility. Hence they are implicitly fixed at value zero, leaving only variable 4 as a free variable. However, setting variable 4 equal to 1 gives a cost (9) which is no better than the incumbent, and therefore this node can be fathomed.




To backtrack, we look for the right-most element without underline. there are none, and therefore the tree is fathomed.

J = {-6, -2, -8}

The current incumbent is therefore optimal.

That is, $X_j = 0$ except for j=6 (found at node 2.)