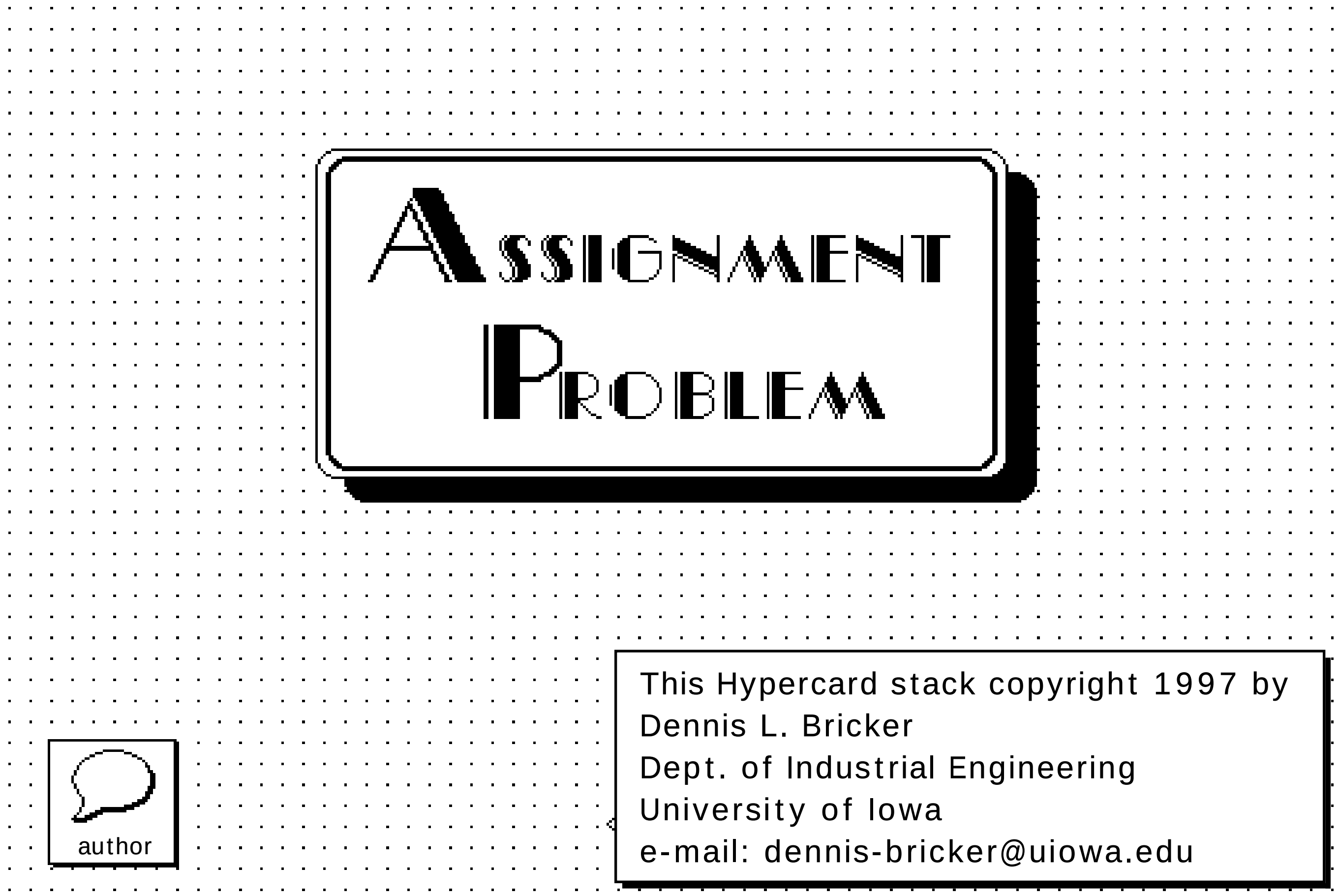

Assignment Problem

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☞ Linear Assignment Problem

☞ Quadratic Assignment Problem

☞ Generalized Assignment Problem

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Assignment Problem

jobs

machines	A	B	C	D	E
1	5	3	2	3	4
2	6	2	1	4	3
3	4	3	3	2	2
4	5	4	2	5	2
5	3	3	2	4	3

cost of completing job

What is the least-cost way of assigning a machine to each of 5 jobs (one job/machine)?

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THE ASSIGNMENT PROBLEM

Each of **n** ***resources*** must be assigned to one of **n** ***activities***, and each activity is assigned exactly one resource.

A cost C_{ij} results if resource i is assigned to activity j.

The objective is to minimize the total cost of assigning every resource to an activity.

Example: assigning jobs to machines in a job-shop

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IP formulation

Let $X_{ij} = \begin{cases} 1 & \text{if resource i is assigned to activity j} \\ 0 & \text{otherwise} \end{cases}$

AP

$$\text{Minimize} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij}$$

subject to

$$\sum_{j=1}^{n} X_{ij} = 1 \text{ for } i=1, 2, \ldots n \leftarrow$$

each resource is assigned to exactly one activity

$$\sum_{i=1}^{n} X_{ij} = 1 \text{ for } j=1, 2, \ldots n \leftarrow$$

each activity is assigned exactly one resource

$$X_{ij} \in \{0,1\} \text{ for all } i \,\&\, j$$

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AP

$$\text{Minimize} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij}$$

subject to

$$\sum_{j=1}^{n} X_{ij} = 1 \text{ for } i=1, 2, \ldots n$$

$$\sum_{i=1}^{n} X_{ij} = 1 \text{ for } j=1, 2, \ldots n$$

$$X_{ij} \in \{0,1\} \text{ for all } i \,\&\, j$$

Note that this is a special case of the transportation problem (with supplies & demands each equal to 1)!

If the restriction that X is 0 or 1 is replaced with a nonnegativity restriction, the LP solution will still be integer!

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AP

Minimize $\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij} X_{ij}$

subject to

$$\sum_{j=1}^{n} X_{ij} = 1 \text{ for } i=1, 2, \ldots n$$

$$\sum_{i=1}^{n} X_{ij} = 1 \text{ for } j=1, 2, \ldots n$$

$$X_{ij} \, \varepsilon \, \{0,1\} \text{ for all } i \text{ \& } j$$

number of basic variables is 2n-1.

number of positive variables is n

Although AP could be solved by the simplex method for TP, all the basic solutions are highly degenerate, which lessens the efficiency of the algorithm.

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Properties of the Assignment Problem

For each i, exactly one assignment $X_{ij}=1$ is made

For each j, exactly one assignment $X_{ij}=1$ is made

Therefore,

If a number **δ** is added to (or subtracted from) every cost in a certain row (or column) of the matrix C,

then every feasible set of assignments will have its cost increased (or decreased) by **δ**,

and the optimal set of assignments remains optimal!

For example, if we add **δ** *to row 1, the total cost is increased by:*

$$\sum_{j=1}^{n} \delta X_{1j} = \delta \sum_{j=1}^{n} X_{1j} = \delta$$ *(independent of X)*

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Properties of the Assignment Problem

If all costs C_{ij} are nonnegative, and if there is a set of assignments with total cost equal to zero,

then

that set of assignments must be optimal.

The "Hungarian Method" solves the assignment problem by adding &/or subtracting quantities in rows &/or columns until an assignment with zero cost is found.

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Example

Four machines are available to process four jobs.

The processing time for each machine/job assignment is as follows:

$$\begin{array}{cc|cccc} & & \multicolumn{4}{c}{\textit{job}} \\ & & 1 & 2 & 3 & 4 \\ \hline & A & 4 & 6 & 5 & 5 \\ \textit{machine} & B & 7 & 4 & 5 & 6 \\ & C & 4 & 7 & 6 & 4 \\ & D & 5 & 3 & 4 & 7 \end{array}$$

What is the assignment (one job per machine) which will minimize total processing time?

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Row reduction

For example, 4 is subtracted from each cost in the first row:

$$\begin{array}{cc|cccc} & & \multicolumn{4}{c}{\text{job}} \\ & & 1 & 2 & 3 & 4 \\ \hline \text{machine} & A & 4 & 6 & 5 & 5 \\ & B & 7 & 4 & 5 & 6 \\ & C & 4 & 7 & 6 & 4 \\ & D & 5 & 3 & 4 & 7 \end{array} \Rightarrow \begin{array}{cc|cccc} & & \multicolumn{4}{c}{\text{job}} \\ & & 1 & 2 & 3 & 4 \\ \hline \text{machine} & A & 0 & 2 & 1 & 1 \\ & B & 3 & 0 & 1 & 2 \\ & C & 0 & 3 & 2 & 0 \\ & D & 2 & 0 & 1 & 4 \end{array}$$

From each row, subtract the smallest cost.
This introduces at least one zero into each row!

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Column reduction

Only column 3 lacks a zero, so only column 3 is reduced:

$$\begin{array}{cc|cccc} & & \multicolumn{4}{c}{\text{job}} \\ & & 1 & 2 & 3 & 4 \\ \hline \text{machine} & A & 0 & 2 & 1 & 1 \\ & B & 3 & 0 & 1 & 2 \\ & C & 0 & 3 & 2 & 0 \\ & D & 2 & 0 & 1 & 4 \end{array} \Rightarrow \begin{array}{cc|cccc} & & \multicolumn{4}{c}{\text{job}} \\ & & 1 & 2 & 3 & 4 \\ \hline \text{machine} & A & 0 & 2 & 0 & 1 \\ & B & 3 & 0 & 0 & 2 \\ & C & 0 & 3 & 1 & 0 \\ & D & 2 & 0 & 0 & 4 \end{array}$$

From each column, subtract the smallest cost.
If a column already has a zero, it is unchanged.
Otherwise, a zero is introduced into the column.

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$$\begin{array}{cc|cccc} & & \multicolumn{4}{c}{\textit{job}} \\ & & 1 & 2 & 3 & 4 \\ \hline & A & \boxed{0} & 2 & 0 & 1 \\ \textit{machine} & B & 3 & \boxed{0} & 0 & 2 \\ & C & 0 & 3 & 1 & \boxed{0} \\ & D & 2 & 0 & \boxed{0} & 4 \end{array}$$

Examining the cost matrix, we can find an assignment with total cost equal to zero:

machine	*job*
A	1
B	2
C	4
D	3

Therefore, this must be an optimal assignment!

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Sometimes, however, one cannot find a zero-cost assignment after row- & column-reduction.

For example: machine C cannot be assigned to both jobs 1 & 4, so one job must be assigned a machine with positive cost

$$\begin{array}{cc|cccc} & & \multicolumn{4}{c}{\textit{job}} \\ & & 1 & 2 & 3 & 4 \\ \hline & A & 4 & 2 & 0 & 1 \\ \textit{machine} & B & 3 & 0 & 0 & 2 \\ & C & \boxed{0} & 3 & 1 & \boxed{0} \\ & D & 2 & 0 & 0 & 4 \end{array}$$

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Hungarian Algorithm

Step 0 Convert to standard form, with
# rows = # columns

Step 1 *Row reduction:* find the smallest cost in each row, and reduce all costs in that row by this amount.

Step 2 *Column reduction:* find the smallest cost in each column, and reduce all costs in the column by this amount.

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Hungarian Algorithm

Step 3 find the minimum number of lines through rows &/or columns necessary to cover all of the zeroes in the cost matrix. If this equals n, STOP.

Step 4 locate the smallest unlined cost, $\bar{c}$. Subtract this cost from all unlined costs, and add to costs at intersections of lines. Return to step 3.

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Justification for step 4:

"Subtract smallest unlined cost $\bar{c}$ from all unlined costs; add to costs at intersections of lines."

is equivalent to

"Subtract $\frac{1}{2}\bar{c}$ from each unlined row & each unlined column.

Add $\frac{1}{2}\bar{c}$ to each lined row and each lined column."

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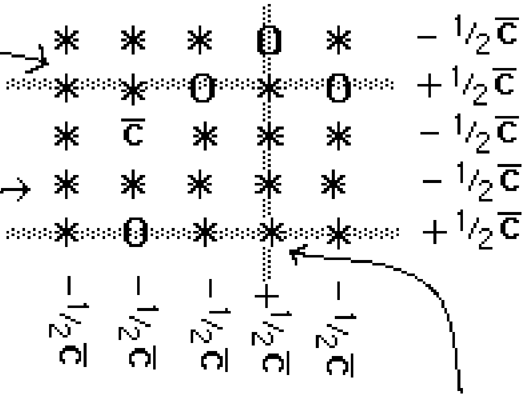
"Subtract $\frac{1}{2}\bar{c}$ from each unlined row & each unlined column.

Add $\frac{1}{2}\bar{c}$ to each lined row and each lined column."

cost with only one line is changed by: $1/2\,\bar{c} - 1/2\,\bar{c}$ i.e., zero

cost with no lines is changed by: $-1/2\,\bar{c} - 1/2\,\bar{c}$ i.e., $-\bar{c}$

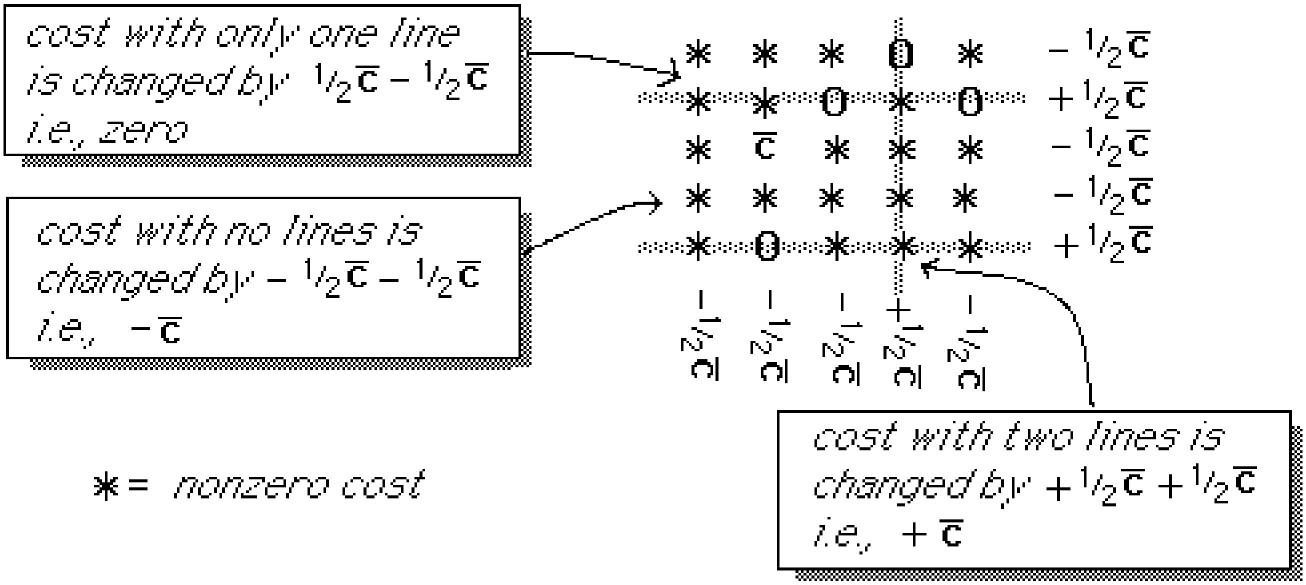


$$\begin{array}{ccccc|c}
\ast & \ast & \ast & 0 & \ast & -1/2\,\bar{c} \\
\ast & \ast & 0 & \ast & 0 & +1/2\,\bar{c} \\
\ast & \bar{c} & \ast & \ast & \ast & -1/2\,\bar{c} \\
\ast & \ast & \ast & \ast & \ast & -1/2\,\bar{c} \\
\ast & 0 & \ast & \ast & \ast & +1/2\,\bar{c} \\
\hline
-1/2\,\bar{c} & -1/2\,\bar{c} & -1/2\,\bar{c} & +1/2\,\bar{c} & -1/2\,\bar{c} &
\end{array}$$

cost with two lines is changed by: $+1/2\,\bar{c} + 1/2\,\bar{c}$ i.e., $+\bar{c}$

$\ast$ = *nonzero cost*

Therefore, step 4 redistributes the zeroes without changing the optimal assignment.



cost with only one line is changed by $\frac{1}{2}\bar{c} - \frac{1}{2}\bar{c}$ i.e., zero

cost with no lines is changed by $-\frac{1}{2}\bar{c} - \frac{1}{2}\bar{c}$ i.e., $-\bar{c}$

cost with two lines is changed by $+\frac{1}{2}\bar{c} + \frac{1}{2}\bar{c}$ i.e., $+\bar{c}$

∗ = nonzero cost

∗ ∗ ∗ 0 ∗ $-\frac{1}{2}\bar{c}$
∗ ∗ 0 ∗ 0 $+\frac{1}{2}\bar{c}$
∗ $\bar{c}$ ∗ ∗ ∗ $-\frac{1}{2}\bar{c}$
∗ ∗ ∗ ∗ ∗ $-\frac{1}{2}\bar{c}$
∗ 0 ∗ ∗ ∗ $+\frac{1}{2}\bar{c}$
$-\frac{1}{2}\bar{c}$ $-\frac{1}{2}\bar{c}$ $-\frac{1}{2}\bar{c}$ $+\frac{1}{2}\bar{c}$ $-\frac{1}{2}\bar{c}$

Row reduction

$$\begin{array}{cc|cccc|} & & \multicolumn{4}{c}{\text{job}} \\ & & 1 & 2 & 3 & 4 \\ \hline & A & 6 & 4 & 5 & 5 \\ \text{machine} & B & 7 & 4 & 5 & 6 \\ & C & 4 & 7 & 6 & 4 \\ & D & 5 & 3 & 4 & 7 \\ \hline \end{array} \Rightarrow \begin{array}{cc|cccc|} & & \multicolumn{4}{c}{\text{job}} \\ & & 1 & 2 & 3 & 4 \\ \hline & A & 2 & 0 & 1 & 1 \\ \text{machine} & B & 3 & 0 & 1 & 2 \\ & C & 0 & 3 & 2 & 0 \\ & D & 2 & 0 & 1 & 4 \\ \hline \end{array}$$

Let's modify the original example somewhat, and repeat the row and column reductions.

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Column reduction

$$\begin{array}{cc|cccc} & & \multicolumn{4}{c}{\text{job}} \\ & & 1 & 2 & 3 & 4 \\ \hline \text{machine} & A & 2 & 0 & 1 & 1 \\ & B & 3 & 0 & 1 & 2 \\ & C & 0 & 3 & 2 & 0 \\ & D & 2 & 0 & 1 & 4 \end{array} \Rightarrow \begin{array}{cc|cccc} & & \multicolumn{4}{c}{\text{job}} \\ & & 1 & 2 & 3 & 4 \\ \hline \text{machine} & A & 2 & 0 & 0 & 1 \\ & B & 3 & 0 & 0 & 2 \\ & C & 0 & 3 & 1 & 0 \\ & D & 2 & 0 & 0 & 4 \end{array}$$

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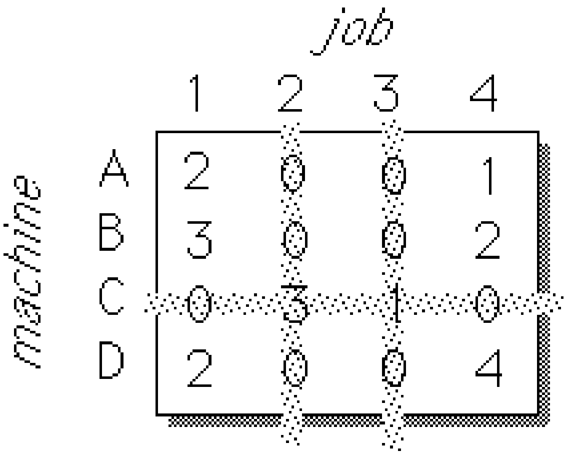
As we saw earlier, there is no zero-cost assignment possible with this matrix.

This can be determined by the fact that the zeroes can be covered with only 3 lines:

$$\begin{array}{cc|cccc} & & \multicolumn{4}{c}{\text{job}} \\ & & 1 & 2 & 3 & 4 \\ \hline \text{machine} & A & 2 & 0 & 0 & 1 \\ & B & 3 & 0 & 0 & 2 \\ & C & 0 & 3 & 1 & 0 \\ & D & 2 & 0 & 0 & 4 \end{array}$$



$$\begin{array}{cc|cccc} & & \multicolumn{4}{c}{\text{job}} \\ & & 1 & 2 & 3 & 4 \\ \hline \text{machine} & A & 2 & 0 & 0 & 1 \\ & B & 3 & 0 & 0 & 2 \\ & C & 0 & 3 & 1 & 0 \\ & D & 2 & 0 & 0 & 4 \end{array}$$

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We therefore perform the reduction in step 4:

Step 4 locate the smallest unlined cost, $\overline{c}$. Subtract this cost from all unlined costs, and add to costs at intersections of lines.

$$\begin{array}{cc|cccc|}
 & & & \text{job} & & \\
 & & 1 & 2 & 3 & 4 \\
\hline
 & A & 2 & 0 & 0 & 1 \\
 & B & 3 & 0 & 0 & 2 \\
\text{machine} & C & 0 & 3 & 1 & 0 \\
 & D & 2 & 0 & 0 & 4 \\
\hline
\end{array}
\Rightarrow
\begin{array}{cc|cccc|}
 & & & \text{job} & & \\
 & & 1 & 2 & 3 & 4 \\
\hline
 & A & 1 & 0 & 0 & 0 \\
 & B & 2 & 0 & 0 & 1 \\
\text{machine} & C & 0 & 4 & 2 & 0 \\
 & D & 1 & 0 & 0 & 3 \\
\hline
\end{array}$$

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The new cost matrix has a zero not covered by a line:

$$\begin{array}{cc|cccc|}
 & & & \text{job} & & \\
 & & 1 & 2 & 3 & 4 \\
\hline
 & A & 1 & 0 & 0 & 0 \\
 & B & 2 & 0 & 0 & 1 \\
\text{machine} & C & 0 & 4 & 2 & 0 \\
 & D & 1 & 0 & 0 & 3 \\
\hline
\end{array}$$

The zeroes now require 4 lines in order to cover all of them!

$$\begin{array}{cc|cccc|}
 & & & \text{job} & & \\
 & & 1 & 2 & 3 & 4 \\
\hline
 & A & 1 & 0 & 0 & 0 \\
 & B & 2 & 0 & 0 & 1 \\
\text{machine} & C & 0 & 4 & 2 & 0 \\
 & D & 1 & 0 & 0 & 3 \\
\hline
\end{array}$$

In fact, there are two different zero-cost assignments, both of them optimal for this problem:

$$\begin{array}{cc|cccc|} & & & \text{job} & & \\ & & 1 & 2 & 3 & 4 \\ \hline & A & 1 & 0 & 0 & \boxed{0} \\ \text{machine} & B & 2 & \boxed{0} & 0 & 1 \\ & C & \boxed{0} & 4 & 2 & 0 \\ & D & 1 & 0 & \boxed{0} & 3 \\ \hline \end{array} \qquad \begin{array}{cc|cccc|} & & & \text{job} & & \\ & & 1 & 2 & 3 & 4 \\ \hline & A & 1 & 0 & 0 & \boxed{0} \\ \text{machine} & B & 2 & 0 & \boxed{0} & 1 \\ & C & \boxed{0} & 4 & 2 & 0 \\ & D & 1 & \boxed{0} & 0 & 3 \\ \hline \end{array}$$