

# Facility Location Problems in the Plane



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*Suppose that we wish to select the location of a single facility, anywhere in the plane, to serve a set of demand points.*

**Given**, for each of demand points  $j=1, 2, \dots, n$ :

$(x_j, y_j)$  coordinates of the point

$\beta_j$  cost per unit volume per unit distance

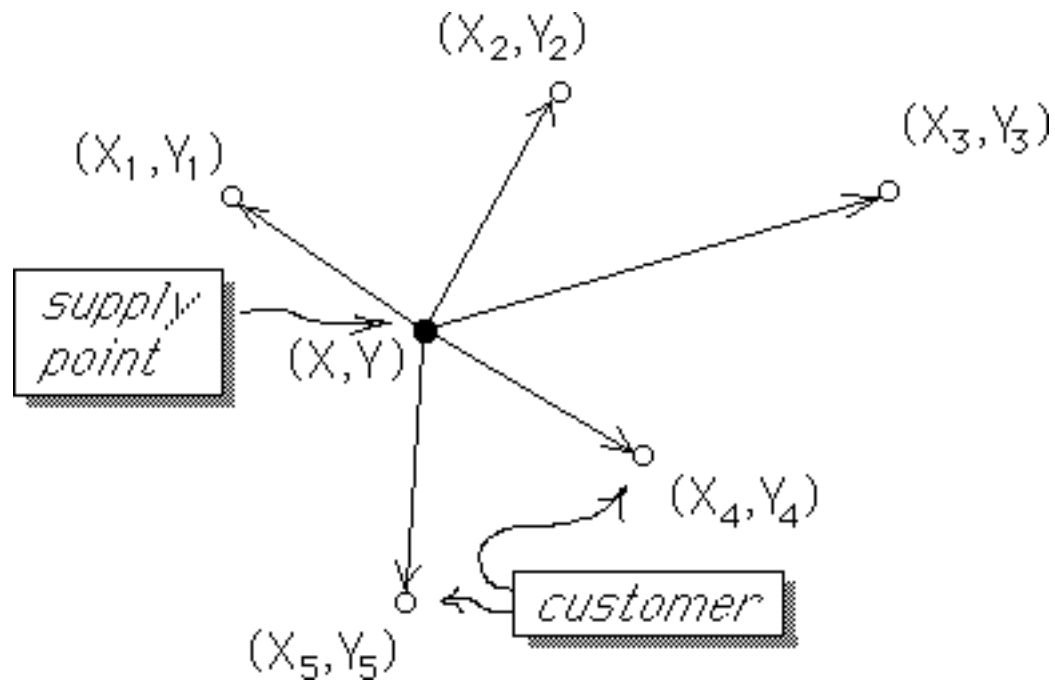
$w_j$  volume of shipments per unit time

**Find** coordinates of the source facility,  $(x, y)$ , which will minimize the total shipping cost per unit time:

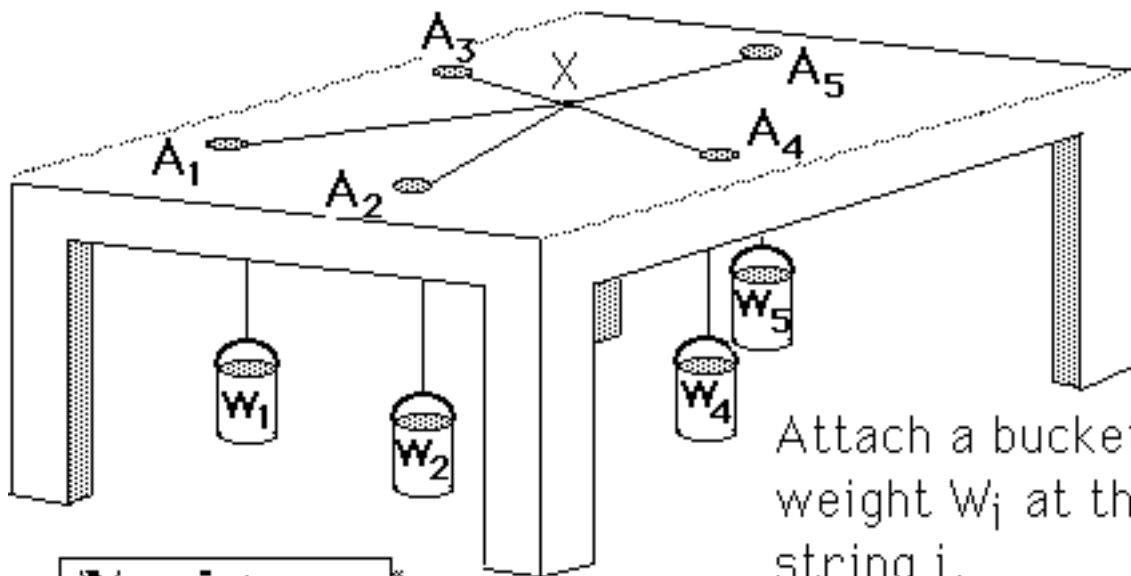
$$\text{Minimize } C(x, y) = \sum_{j=1}^n \beta_j w_j \sqrt{(x-x_j)^2 + (y-y_j)^2}$$

*assuming "straight-line";  
Euclidean distances!*

**Weber's  
Problem**

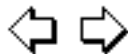


Tie together in a knot ("X") n strings of equal length L



Attach a bucket with weight  $W_i$  at the end of string  $i$ .

**Varignon  
Frame**



Details

**Theorem**

The function:

$$C(x,y) = \sum_{j=1}^n \beta_j w_j \sqrt{(x-x_j)^2 + (y-y_j)^2}$$

is convex in  $(x,y)$

A *necessary* condition for  $(X^*, Y^*)$  to minimize

$$C(x, y) = \sum_{j=1}^n \beta_j w_j \sqrt{(x - x_j)^2 + (y - y_j)^2}$$

is

$$\begin{cases} \frac{\partial}{\partial X} C(X^*, Y^*) = 0 \\ \frac{\partial}{\partial Y} C(X^*, Y^*) = 0 \end{cases}$$

That is,  $(X^*, Y^*)$  should be a "stationary point" of the function  $C$ .

This condition yields the equations

$$\left\{ \begin{array}{l} \sum_{j=1}^n \frac{\beta_j w_j (X^* - x_j)}{\sqrt{(X^* - x_j)^2 + (Y^* - y_j)^2}} = 0 \\ \sum_{j=1}^n \frac{\beta_j w_j (Y^* - y_j)}{\sqrt{(X^* - x_j)^2 + (Y^* - y_j)^2}} = 0 \end{array} \right.$$

which, unfortunately, we cannot solve analytically for the values of  $X^*$  and  $Y^*$ !

For convenience, define a distance function for each  $j$ :

$$d_j(X, Y) = \sqrt{(X - x_j)^2 + (Y - y_j)^2}$$

Necessary  
conditions  
for  
optimality

$$\left\{ \begin{array}{l} \sum_{j=1}^n \frac{\beta_j w_j (X^* - x_j)}{d_j(X^*, Y^*)} = 0 \\ \sum_{j=1}^n \frac{\beta_j w_j (Y^* - y_j)}{d_j(X^*, Y^*)} = 0 \end{array} \right.$$



*Rearrange terms:*

$$\left\{ \begin{array}{l} X^* \sum_{j=1}^n \frac{\beta_j w_j}{d_j(X^*, Y^*)} = \sum_{j=1}^n \frac{\beta_j w_j X_j}{d_j(X^*, Y^*)} \\ Y^* \sum_{j=1}^n \frac{\beta_j w_j}{d_j(X^*, Y^*)} = \sum_{j=1}^n \frac{\beta_j w_j Y_j}{d_j(X^*, Y^*)} \end{array} \right.$$

*Necessary  
Conditions  
for the  
Optimality  
of  $(X^*, Y^*)$*

*Note:  $X^*$  and  $Y^*$   
actually appear  
on both sides of  
the equations!*

$$\left\{ \begin{array}{l} X^* = \frac{\sum_{j=1}^n \frac{\beta_j w_j X_j}{d_j(X^*, Y^*)}}{\sum_{j=1}^n \frac{\beta_j w_j}{d_j(X^*, Y^*)}} \\ Y^* = \frac{\sum_{j=1}^n \frac{\beta_j w_j Y_j}{d_j(X^*, Y^*)}}{\sum_{j=1}^n \frac{\beta_j w_j}{d_j(X^*, Y^*)}} \end{array} \right.$$

We will use a  
"successive  
substitution"  
method using  
these equations  
to find  $X^*$  &  $Y^*$

$$\left\{ \begin{array}{l} X^* = \frac{\sum_{j=1}^n \frac{\beta_j w_j X_j}{d_j(X^*, Y^*)}}{\sum_{j=1}^n \frac{\beta_j w_j}{d_j(X^*, Y^*)}} \\ Y^* = \frac{\sum_{j=1}^n \frac{\beta_j w_j Y_j}{d_j(X^*, Y^*)}}{\sum_{j=1}^n \frac{\beta_j w_j}{d_j(X^*, Y^*)}} \end{array} \right.$$

Suppose, at iteration #k, we have an approximate solution  $(X^k, Y^k)$ .

We obtain an improved approximate solution  $(X^{k+1}, Y^{k+1})$  by

$$X^{k+1} = \frac{\sum_{j=1}^n \frac{\beta_j w_j x_j}{d_j(X^k, Y^k)}}{\sum_{j=1}^n \frac{\beta_j w_j}{d_j(X^k, Y^k)}} \quad \& \quad Y^{k+1} = \frac{\sum_{j=1}^n \frac{\beta_j w_j y_j}{d_j(X^k, Y^k)}}{\sum_{j=1}^n \frac{\beta_j w_j}{d_j(X^k, Y^k)}}$$

## Weiszfeld Algorithm

Starting with an initial "guess"  $(X^0, Y^0)$ , we will generate a sequence of approximate solutions,  $(X^1, Y^1)$ ,  $(X^2, Y^2)$ ,  $(X^3, Y^3)$ , .... which converge to the optimal facility location  $(X^*, Y^*)$ .

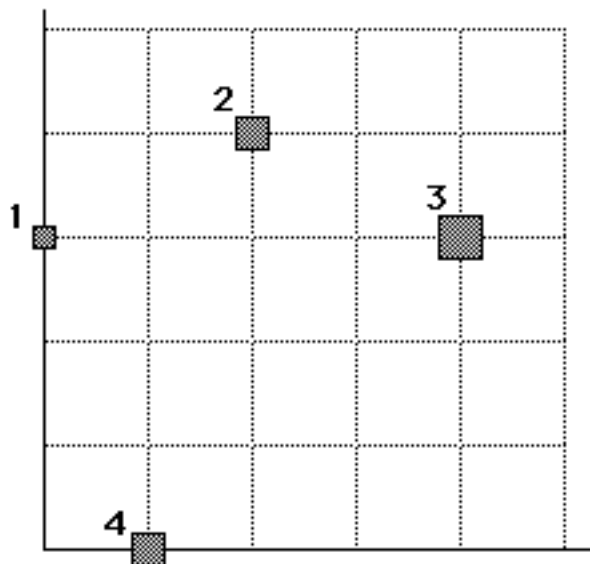
We terminate the method when two successive approximate solutions are "close enough",

i.e.,

$$|X^{k+1} - X^k| + |Y^{k+1} - Y^k| < \epsilon \approx 0$$

**Example**

Customer	1	2	3	4
Location	(0,3)	(2,4)	(4,3)	(1,0)
Rqmt.(Ton/wk)	1	2	3	2



*Cost/ton-mile  
is same for all  
customers*

*Where should a supply  
facility be located so  
that total shipping cost  
per week is minimized?*

**Starting Point**

*A good starting point is the centroid, i.e., the weighted average of the customer coordinates.*

$$X^0 = \frac{\sum_{j=1}^4 \beta_j W_j X_j}{\sum_{j=1}^4 \beta_j W_j} \quad \& \quad Y^0 = \frac{\sum_{j=1}^4 \beta_j W_j Y_j}{\sum_{j=1}^4 \beta_j W_j}$$

Customer	1	2	3	4
Location	(0,3)	(2,4)	(4,3)	(1,0)
Rqmt.	1	2	3	2

$$\beta_j = 1 \quad \forall j$$

$$X^0 = \frac{\sum_{j=1}^4 \beta_j w_j X_j}{\sum_{j=1}^4 \beta_j w_j} = \frac{1 \times 0 + 2 \times 2 + 3 \times 4 + 2 \times 1}{1 + 2 + 3 + 2} = 2.25$$

$$Y^0 = \frac{\sum_{j=1}^4 \beta_j w_j Y_j}{\sum_{j=1}^4 \beta_j w_j} = \frac{1 \times 3 + 2 \times 4 + 3 \times 3 + 2 \times 1}{1 + 2 + 3 + 2} = 2.5$$



*Now compute distance from  $(X^0, Y^0)$  to each customer:*

$$d_1 = \sqrt{\left(\frac{9}{4} - 0\right)^2 + \left(\frac{5}{2} - 3\right)^2} \approx 2.305$$

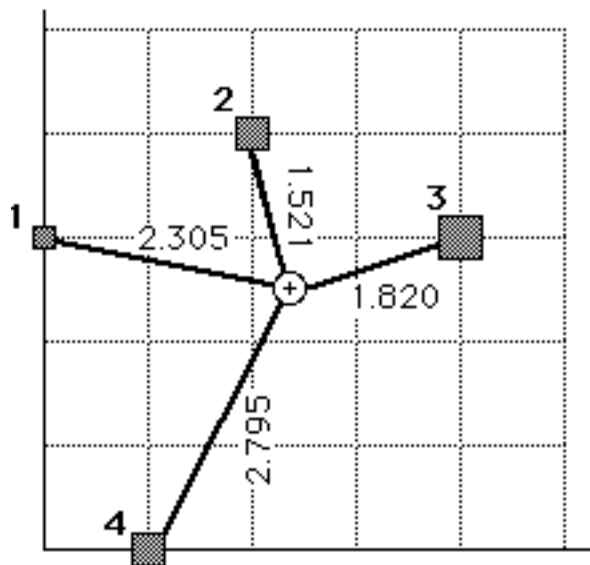
$$d_2 = \sqrt{\left(\frac{9}{4} - 2\right)^2 + \left(\frac{5}{2} - 4\right)^2} \approx 1.521$$

$$d_3 = \sqrt{\left(\frac{9}{4} - 4\right)^2 + \left(\frac{5}{2} - 3\right)^2} \approx 1.820$$

$$d_4 = \sqrt{\left(\frac{9}{4} - 1\right)^2 + \left(\frac{5}{2} - 0\right)^2} \approx 2.795$$

### *Shipping Cost:*

$$\begin{aligned} &1(2.305) + 2(1.521) \\ &+ 3(1.82) + 2(2.795) \\ &= 16.397 \end{aligned}$$



*Apply our successive substitution method to (we hope!) obtain a better approximate solution:*

$$X^1 = \frac{\sum_j \frac{w_j X_j}{d_j}}{\sum_j \frac{w_j}{d_j}} = \frac{\frac{1 \times 0}{d_1} + \frac{2 \times 2}{d_2} + \frac{3 \times 4}{d_3} + \frac{2 \times 1}{d_4}}{\frac{1}{d_1} + \frac{2}{d_2} + \frac{3}{d_3} + \frac{2}{d_4}} \approx \frac{10.373}{4.113} = 2.522$$

$$Y^1 = \frac{\sum_j \frac{w_j Y_j}{d_j}}{\sum_j \frac{w_j}{d_j}} = \frac{\frac{1 \times 3}{d_1} + \frac{2 \times 4}{d_2} + \frac{3 \times 3}{d_3} + \frac{2 \times 0}{d_4}}{\frac{1}{d_1} + \frac{2}{d_2} + \frac{3}{d_3} + \frac{2}{d_4}} \approx \frac{11.506}{4.113} = 2.798$$

*distance to  
customer j*

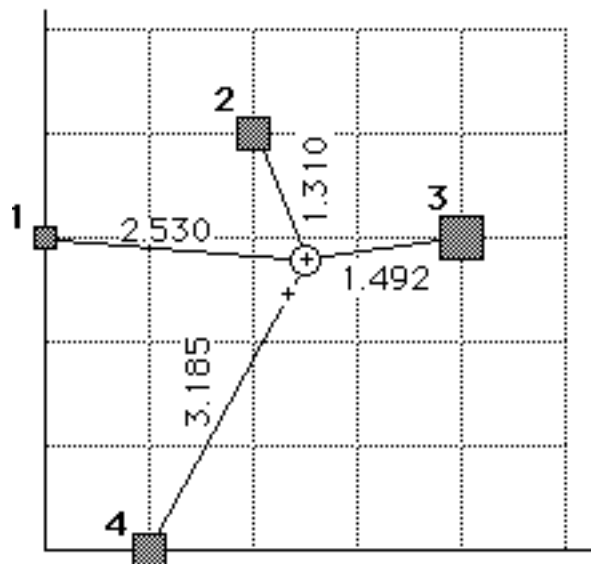
$$d_j = \sqrt{[2.522 - X_j]^2 + [2.798 - Y_j]^2}$$

$$d_1 = 2.530$$

$$d_2 = 1.310$$

$$d_3 = 1.492$$

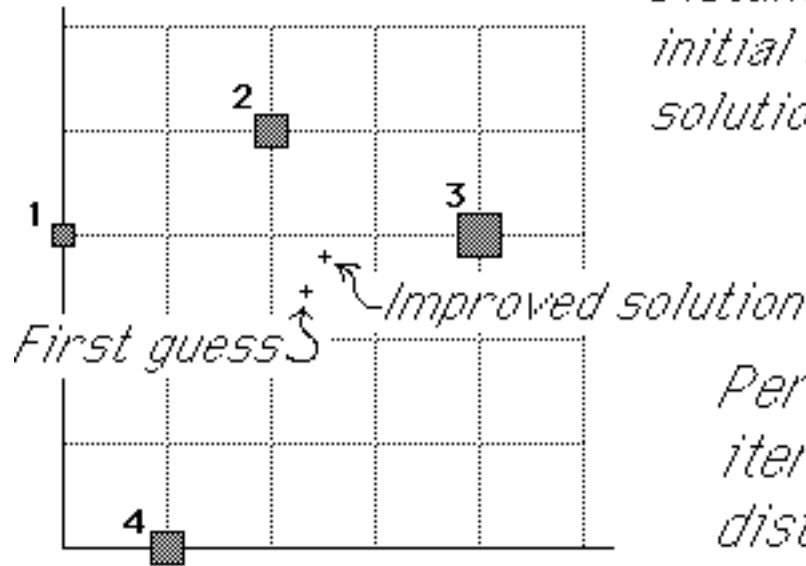
$$d_4 = 3.185$$



*Shipping cost*

$$1(2.53) + 2(1.31) + 3(1.492) + 2(3.185) = 15.996 < 16.397$$

*reduction of 2.4%*



*Distance between  
initial and improved  
solution:*

0.421

*Perform additional  
iterations, until  
distance moved is  
"sufficiently small"*

Iteration # 1
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Facility location at X= 2.25, Y= 2.5

Distances to demand pts:

i	1	2	3	4
D[i]	2.30489	1.52069	1.82003	2.79508
WT[i]×D[i]	2.30489	3.04138	5.46008	5.59017

Total cost is 16.3965

New location is at X= 2.41659, Y= 2.79785

Rectilinear distance moved is 0.464436

Iteration # 2
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Facility location at X= 2.41659, Y= 2.79785

Distances to demand pts:

i	1	2	3	4
D[i]	2.42503	1.27229	1.59626	3.13603
WT[i]×D[i]	2.42503	2.54457	4.78879	6.27206

Total cost is 16.0305

New location is at X= 2.51012, Y= 2.92419

Rectilinear distance moved is 0.21987

Iteration # 3
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Facility location at X= 2.51012, Y= 2.92419

Distances to demand pts:

i	1	2	3	4
D[i]	2.51126	1.19063	1.49181	3.2911
WT[i]×D[i]	2.51126	2.38126	4.47542	6.5822

Total cost is 15.9501

New location is at X= 2.55738, Y= 2.96949

Rectilinear distance moved is 0.0925647



Iteration # 4
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Facility location at X= 2.55738, Y= 2.96949

Distances to demand pts:

i	1	2	3	4
D[i]	2.55757	1.1716	1.44294	3.3531
WT[i]×D[i]	2.55757	2.34319	4.32881	6.7062

Total cost is 15.9358

New location is at X= 2.58231, Y= 2.98276

Rectilinear distance moved is 0.0381954

Iteration # 5
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Facility location at X= 2.58231, Y= 2.98276

Distances to demand pts:

i	1	2	3	4
D[i]	2.58237	1.17212	1.4178	3.37647
WT[i]×D[i]	2.58237	2.34424	4.25339	6.75294

Total cost is 15.9329

New location is at X= 2.59667, Y= 2.98528

Rectilinear distance moved is 0.0168786

Iteration # 6
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Facility location at X= 2.59667, Y= 2.98528

Distances to demand pts:

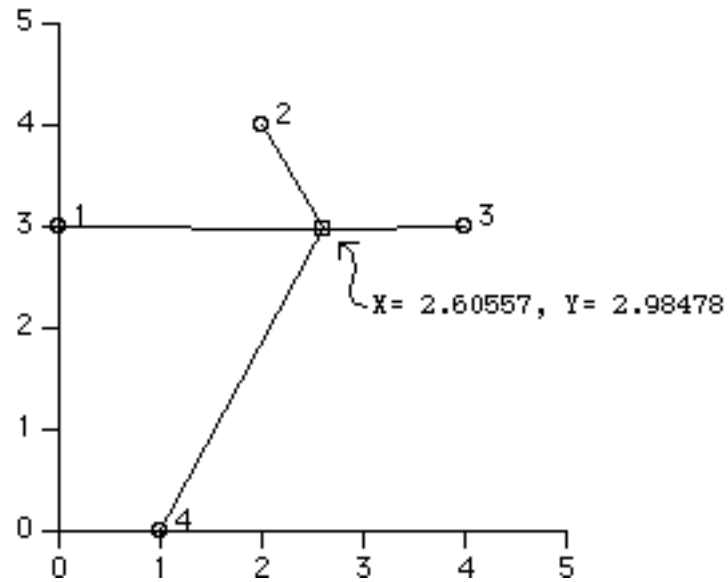
i	1	2	3	4
D[i]	2.59671	1.17715	1.40341	3.38544
WT[i]×D[i]	2.59671	2.35429	4.21023	6.77089

Total cost is 15.9321

New location is at X= 2.60557, Y= 2.98478

Rectilinear distance moved is 0.0094046 < 0.01 (stopping criterion)

## The Optimal Location for the Supply Facility



## Path Followed by the Successive Substitution Method

