Stochastic Transportation Problem

author

Example

Consider a transportation problem in which some of the demands are random variables:

SOURCES	DESTINATIONS 1	2	3	4	supply
1	2	3	11	7	6
2	1	1	6	1	1
3	5	8	15	9	10
demand	7	5	D_3	D_4	

The demands at destinations #3 & 4 are ***random*** with known probability distributions:

d	$P\{D_3=d\}$
1	1/3
3	1/3
5	1/3

d	$P\{D_4=d\}$
0	1/4
4	3/4

Shipments X_{ij} must be selected ***before*** the values of D_3 and D_4 are known!

If, after making the shipments, the demand is different from the quantity shipped, we must act so as to compensate for the difference:

- if demand exceeds amount shipped, the amount short must be obtained at high cost (e.g., by purchasing locally, or shipment by air, etc.)
- if demand is less than amount shipped, the excess must be stored, sold at a loss, or otherwise disposed of.

In this case, assume penalties of \$9 and \$3 per unit short at destinations #3 and 4, respectively, but no cost incurred by excess supplies.

We wish to minimize the sum of the

- *shipping costs*
- *expected shortage penalties*

d	$P\{D_3=d\}$
1	$1/3$
3	$1/3$
5	$1/3$

d	$P\{D_4=d\}$
0	$1/4$
4	$3/4$

Six possible outcomes:

k	1	2	3	4	5	6
D_3^k	1	3	5	1	3	5
D_4^k	0	0	0	4	4	4
p^k	$1/12$	$1/12$	$1/12$	$1/4$	$1/4$	$1/4$

Joint probabilities assume demands are independent random variables!

Define:

"First-stage variables"

X_{ij} = quantity shipped from source i to destination j

"Second-stage variables"

Y_j^{k+} = surplus at destination j if outcome k occurs

i.e., amount to be disposed of.

Y_j^{k-} = shortage at destination j if outcome k occurs

i.e., amount to be purchased locally.

Equivalent Deterministic LP Model

$$\text{Minimize} \sum_{i=1}^{3}\sum_{j=1}^{4} C_{ij}X_{ij} + \sum_{k=1}^{6}\sum_{j=3}^{4} P^k E_j Y_j^{k-}$$

penalty/unit shortage

subject to

$$\sum_{j=1}^{4} X_{ij} \le S_i \;,\; i=1,\,2,\,3$$

$$\sum_{i=1}^{3} X_{ij} \ge D_j \;,\; j=1 \;\&\; 2$$

$$\sum_{i=1}^{3} X_{ij} + Y_j^{k-} - Y_j^{k+} = D_j^k \;,\; j=3 \;\&\; 4,\; k=1,\ldots 6$$

$$X_{ij} \ge 0,\; Y_j^{k+} \ge 0,\; Y_j^{k-} \ge 0$$

Size of the LP

Variables: 12 X's
24 Y's
36 total

Constraints: 17

j=	X_{1j} 1	2	3	4	X_{2j} 1	2	3	4	X_{3j} 1	2	3	4	$Y_j^{1\pm}$ j=3 −	+	j=4 −	+	$Y_j^{2\pm}$ j=3 −	+	j=4 −	+	$Y_j^{3\pm}$ j=3 −	+	j=4 −	+	$Y_j^{4\pm}$ j=3 −	+	j=4 −	+	$Y_j^{5\pm}$ j=3 −	+	j=4 −	+	$Y_j^{6\pm}$ j=3 −	+	j=4 −	+	
1	1	1	1	1																																	6
2					1	1	1	1																													1
3									1	1	1	1																									10
4	1				1				1																												7
5		1				1				1																											5
6			1				1				1		1	−1																							1
7				1				1				1			1	−1																					0
8			1				1				1						1	−1																			3
9				1				1				1							1	−1																	0
10			1				1				1										1	−1															5
11				1				1				1											1	−1													0
12			1				1				1														1	−1											1
13				1				1				1															1	−1									4
14			1				1				1																		1	−1							3
15				1				1				1																			1	−1					4
16			1				1				1																						1	−1			5
17				1				1				1																							1	−1	4

Constraint Coefficient Matrix

The Coefficient Matrix of the constraints is not a node-arc incidence matrix, but does contain only ±1 & 0.

Can we manipulate the rows to obtain a node-arc incidence matrix, with each column containing a ±1 pair?

Perform the following row operations in the sequence indicated:

$R_{17} \leftarrow R_{17} - R_{15}$, *i.e., subtract row 15 from row 17*
$R_{16} \leftarrow R_{16} - R_{14}$, *i.e., subtract row 14 from row 16,*
$R_{15} \leftarrow R_{15} - R_{13}$, *i.e., subtract row 13 from row 15,*
etc.

$R_i \leftarrow R_i - R_{i-2}$, i=17, 16, 15, 14, ... 8

Next, negate all but Rows 1, 2, & 3.

	X_{1j}				X_{2j}				X_{3j}				$Y_j^{1\pm}$				$Y_j^{2\pm}$				$Y_j^{3\pm}$				$Y_j^{4\pm}$				$Y_j^{5\pm}$				$Y_j^{6\pm}$					
j=	1	2	3	4	1	2	3	4	1	2	3	4	j=3 −	j=3 +	j=4 −	j=4 +	j=3 −	j=3 +	j=4 −	j=4 +	j=3 −	j=3 +	j=4 −	j=4 +	j=3 −	j=3 +	j=4 −	j=4 +	j=3 −	j=3 +	j=4 −	j=4 +	j=3 −	j=3 +	j=4 −	j=4 +		
1	1	1	1	1																																	≤	6
2					1	1	1	1																													≤	1
3									1	1	1	1																									≤	10
4	-1				-1				-1																												=	-7
5		-1				-1				-1																											=	-5
6			-1				-1				-1		-1	+1																							=	-1
7				-1				-1				-1			-1	+1																					=	0
8													+1	-1			-1	+1																			=	-2
9															+1	-1			-1	+1																	=	0
10																	+1	-1			-1	+1															=	-2
11																			+1	-1			-1	+1													=	0
12																					+1	-1			-1	+1											=	4
13																							+1	-1			-1	+1									=	-4
14																									+1	-1			-1	+1							=	-2
15																											+1	-1			-1	+1					=	0
16																													+1	-1			-1	+1			=	-2
17																															+1	-1			-1	+1	=	0

...almost a node-arc incidence matrix!

We next change rows 1, 2, & 3 to equations by adding slack variables.

Each column now contains one ±1 pair except for the last seven (the three slack variables added to rows 1 to 3, together with the Y variables for the last (sixth) outcome). These seven columns each contain either a +1 or a −1 only.

The transformation to a node-arc incidence matrix may now be completed by appending a new (redundant) row, obtained by negating the sum of Rows #1 through #17.

Columns already having a ± 1 pair will have a sum of zero, while columns having only a + 1 or a − 1 will have the pair completed.

	X_{1j}				X_{2j}				X_{3j}				$Y_j^{1\pm}$				$Y_j^{2\pm}$				$Y_j^{3\pm}$				$Y_j^{4\pm}$				$Y_j^{5\pm}$			
j=	1	2	3	4	1	2	3	4	1	2	3	4	j=3		4		j=3		4		j=3		4		j=3		4		j=3		4	
													−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+
1	1	1	1	1																												
2					1	1	1	1																								
3									1	1	1	1																				
4	-1				-1				-1																							
5		-1				-1				-1																						
6			-1				-1				-1		-1	+1																		
7				-1				-1				-1			-1	+1																
8													+1	-1			-1	+1														
9															+1	-1			-1	+1												
10																	+1	-1			-1	+1										
11																			+1	-1			-1	+1								
12																					+1	-1			-1	+1						
13																							+1	-1			-1	+1				
14																									+1	-1			-1	+1		
15																											+1	-1			-1	+1
16																													+1	-1		
17																															+1	-1
18																																

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row #	$Y_j^{6\pm}$ j=3 −	j=3 +	j=4 −	j=4 +	S_1	S_2	S_3	
1					1			6
2						1		1
3							1	10
4								-7
5								-5
6								-1
7								0
8								-2
9								0
10								-2
11								0
12								4
13								-4
14								-2
15								0
16	-1	+1						-2
17			-1	+1				0
18	+1	-1	+1	-1	-1	-1	-1	4

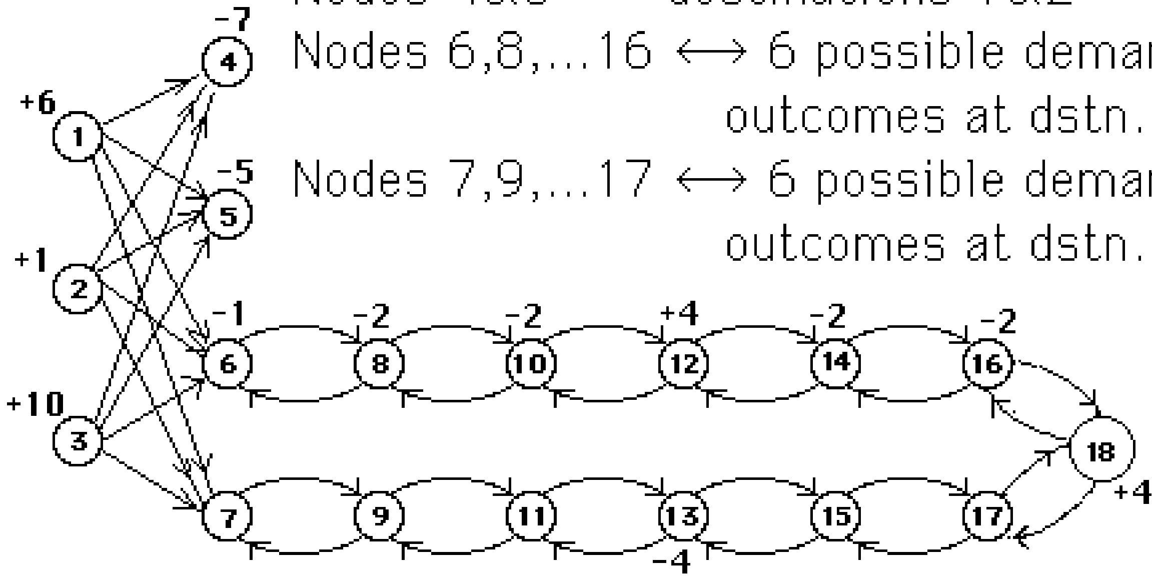
Let's now draw the network, with a node for each row, an arc for each column

Nodes 1,2,&3 ⟷ sources 1,2,&3

Nodes 4&5 ⟷ destinations 1&2

Nodes 6,8,...16 ⟷ 6 possible demand outcomes at dstn. #3

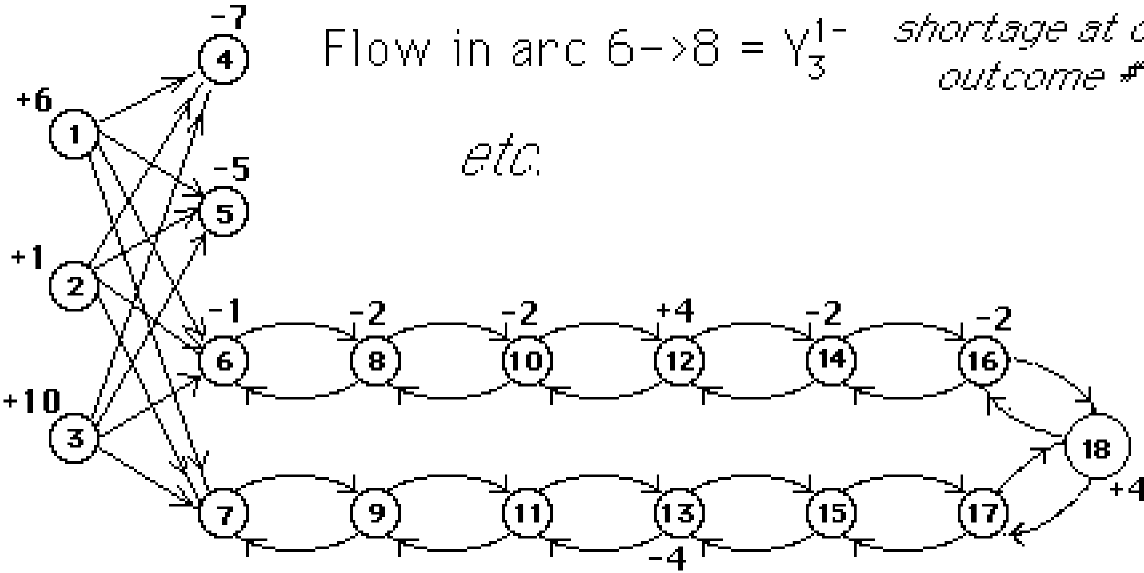
Nodes 7,9,...17 ⟷ 6 possible demand outcomes at dstn. #4



Flow in arc 8->6 = Y_3^{1+} *surplus at dstn 3 if outcome #1*

Flow in arc 6->8 = Y_3^{1-} *shortage at dstn 3 if outcome #1*

etc.



Optimal Solution

Ship to satisfy

minimum demand (1) at dstn. 3

maximum demand (4) at dstn 4