SIMPLE PLANT LOCATION PROBLEM

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Given: M candidate locations, N customers

F_i = fixed cost of establishing a plant at site i, i=1,2,...M

SPL

C_{ij} = cost of supplying all demand of customer j from plant i, j=1,2,...N

The Problem: Select a set of plant locations and allocation of customers to plants so as to minimize the total cost.

Note: there are no capacity constraints for a plant which has been selected, and the number of plants is not specified (unlike p-median problem)

ILP models of the SPL problem

Define variables:

$$Y_i = \begin{cases} 1 & \text{if plant site } i \text{ is selected} \\ 0 & \text{otherwise} \end{cases}$$

$$X_{ij} = \begin{cases} 1 & \text{if plant } i \text{ serves all demand of customer } j \\ 0 & \text{otherwise} \end{cases}$$

Model #1

$$\text{Minimize} \sum_{i=1}^{M} \sum_{j=1}^{N} C_{ij} X_{ij} + \sum_{i=1}^{M} F_i Y_i$$

$$\text{s.t.} \sum_{i=1}^{M} X_{ij} = 1 \quad \forall\ j=1,\ldots N$$

$$X_{ij} \le Y_i \quad \forall\ i\&j$$

$$Y_i \in \{0,1\},\ X_{ij} \ge 0 \quad \forall\ i\&j$$

Model #2

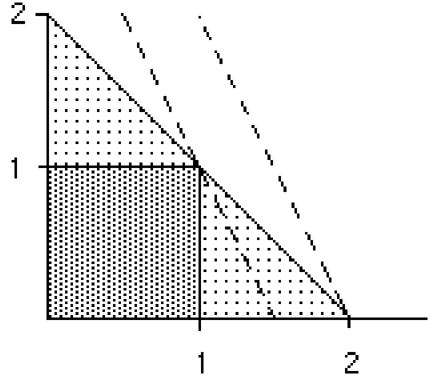
Replace constraints $X_{ij} \le Y_i \quad \forall\ i\&j$

with aggregated constraints

$$\sum_{j=1}^{N} X_{ij} \le N Y_i \quad \forall\ i$$

Models #1 & #2 are equivalent, in that the feasible solution sets are identical....

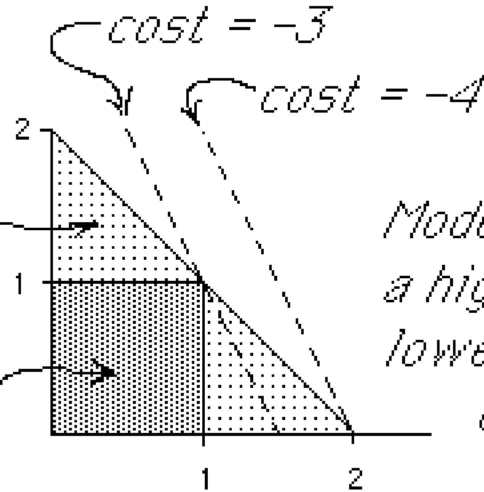
But-- their LP relaxations (i.e., replacing $Y_i \in \{0,1\}$ with $0 \le Y_i \le 1$) are not!



Example

Minimize $-2X_{i1} - X_{i2}$

feasible set for
$X_{i1} + X_{i2} \leq 2$



feasible set for
$X_{i1} \leq 1$
$X_{i2} \leq 1$

Model #1 provides a higher, "better" lower bound on the optimum!

Model #2 is more "compact", and the LP relaxation is easier to solve.

LP Relaxation of Model #2

At the LP optimum,

$$\sum_{j=1}^{N} X_{ij} \leq N Y_i \quad \forall \, \mathbf{i}$$

is "tight",

i.e., $Y_i = \frac{1}{N} \sum_{j=1}^{N} X_{ij}$

Eliminate Y_i **Minimize** $\sum_{i=1}^{M} \sum_{j=1}^{N} C_{ij} X_{ij} + \sum_{i=1}^{M} \frac{1}{N} F_i \sum_{j=1}^{N} X_{ij}$

$$\Longrightarrow \begin{cases} \textbf{Minimize } \sum_{i=1}^{M} \sum_{j=1}^{N} \left[C_{ij} + \frac{F_i}{N} \right] X_{ij} \\ \textbf{s.t. } \sum_{i=1}^{M} X_{ij} = 1 \quad \forall \, \mathbf{j=1, \ldots N} \\ X_{ij} \geq 0 \quad \forall \, \mathbf{i\&j} \end{cases}$$

The solution is $$X_{ij}^* = \begin{cases} 1 & \text{if } C_{ij} + \frac{F_i}{N} \leq C_{kj} + \frac{F_k}{N} \ \forall i \\ 0 & \text{otherwise} \end{cases}$$

with objective value $$\sum_{j=1}^{N} \min_i \left\{ C_{ij} + \frac{F_i}{N} \right\}$$

Although not a strong bound, this is easily computed:

APL

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+/ ⌊⌿ C + ⍉(⌽⍴C)⍴F÷N
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4 = M = # potential plant sites
8 = N = # demand points

Costs

i \ j=	1	2	3	4	5	6	7	8	F
1	4	6	8	9	5	4	3	0	140
2	10	5	10	0	8	10	9	9	120
3	3	5	7	9	4	5	2	3	177
4	8	6	4	7	5	10	8	8	128
D	98	12	7	33	49	33	87	78	

Weak LP Relaxation of Simple Plant Location Problem

The Matrix C + (F÷N)

from \ to	1	2	3	4	5	6	7	8
1	144	146	148	149	145	144	143	140
2	130	125	130	120	128	130	129	129
3	180	182	184	186	181	182	179	180
4	136	134	132	135	133	138	136	136

The LP bound is found by summing the minima in each column

Lower bound provided by weak LP relaxation = 1031.38

Model #3

Minimize $\sum_{i=1}^{M} f_i(X_{i1}, X_{i2}, \cdots X_{iN})$

subject to $\sum_{i=1}^{M} X_{ij} = 1 \quad \forall j=1,2,\ldots N$

$X_{ij} \geq 0 \ \forall \ i\&j$

continuous variables only, but objective is discontinuous

where

$$f_i(X_{i1}, X_{i2}, \cdots X_{iN}) = \begin{cases} 0 \text{ if } \sum_{j=1}^{N} X_{ij} = 0 \\ F_i + \sum_{j=1}^{N} C_{ij} X_{ij} \text{ otherwise} \end{cases}$$

Surrogate Constraint

Define a ***surrogate multiplier*** for each constraint: U_j, $j=1,\cdots N$; $\sum_j U_j = 1$

Form a linear combination of the constraints

$$\left.\begin{array}{c} U_1 \times \sum_i X_{i1} = U_1 \times 1 \\ \vdots \\ U_N \times \sum_i X_{iN} = U_N \times 1 \end{array}\right\} \Rightarrow \sum_j U_j \sum_i X_{ij} = \sum_j U_j \Rightarrow \sum_j \sum_i U_j X_{ij} = 1$$

This ***surrogate constraint*** is implied by the original set of constraints, but is less restrictive.

Surrogate Relaxation

We replace the original constraints of Model #3 with the single surrogate constraint:

$$\text{Minimize} \sum_{i=1}^{M} f_i(X_{i1}, X_{i2}, \cdots X_{iN})$$

$$\text{subject to} \quad \sum_j \sum_i U_j X_{ij} = 1$$

$$X_{ij} \geq 0 \ \forall \ i\&j$$

Because the objective function is ***concave***, the theory of nonlinear programming assures us that an extreme point of the feasible region (i.e., a ***basic*** solution) is optimal, so only a single variable is $\neq 0$.

For example,

$$X_{ij} = \begin{cases} 1/U_q & \text{if } i=p,\ j=q \\ 0 & \text{otherwise} \end{cases}$$

with cost $F_p + C_{pq} \times 1/U_q$

for some p and q.

Therefore, we can solve the surrogate relaxation by enumerating the MxN basic solutions, and selecting the least cost solution:

$$\mathbf{S(U)} = \underset{i,j}{\mathbf{minimum}} \left\{ \mathbf{F}_i + \mathbf{C}_{ij} / \mathbf{U}_j \right\}$$

Because the optimal solution of the original SPL problem is feasible in this surrogate relaxation,

$\mathbf{S(U)} \leq$ optimum of SPL problem

for all $U = (U_1, U_2, \ldots U_N)$

Surrogate Dual Problem

Since for each U, S(U) gives us a lower bound on the SPL optimal value,
select the surrogate multipliers U to give us the "best", i.e., greatest lower bound:

$$\hat{S} = \text{maximum} \quad S(U)$$
$$\text{s.t.} \sum_j U_j = 1$$

Use of Surrogate Dual bound in a Branch-&-Bound algorithm

Given a value **V** (e.g., the incumbent solution), we can fathom a subproblem if its surrogate dual value $\hat{S}$ exceeds V, and this may be tested without explicitly computing $\hat{S}$:

$$\hat{S} \geq V \Longleftrightarrow \exists\ U=(U_1,\cdots U_N) \text{ such that } \begin{cases} V \leq F_i + C_{ij}/U_j & \forall\ i\&j \\ \sum_j U_j = 1 \end{cases}$$

Assuming $F_i < V$, this is equivalent to

$$\begin{cases} U_j \le \dfrac{C_{ij}}{V - F_i} \ \forall i \& j \\ \sum_j U_j = 1 \end{cases}$$

which clearly has a solution if and only if the least upper bounds of U_j, $j=1,\ldots N$, have a sum ≥ 1:

$$\hat{S} \ge V \iff \sum_j \min_i \left\{ \frac{C_{ij}}{V - F_i} \right\} \ge 1$$

$$\frac{C_{ij}}{V - F_i}$$

0.44	0.08081	0.06285	0.3333	0.275	(0.1481)	0.2929	(0)
1.076	(0.06586)	0.07684	(0)	0.4303	0.3622	0.8595	0.7706
(0.3443)	0.07026	0.05738	0.3478	(0.2295)	0.1932	(0.2037)	0.274
0.8682	0.07973	(0.03101)	0.2558	0.2713	0.3654	0.7708	0.691

Sum: $$\sum_j \min_i \left\{ \frac{C_{ij}}{V - F_i} \right\} = 1.023$$

The conclusion of the comparison test is:

$\hat{S} \geq V$ (= 1031)

By any of several methods, the equation

$$\sum_j \min_i \left\{ \frac{C_{ij}}{- F_i} \right\} = 1$$

may easily be solved for $\hat{S}$ if the actual value of $\hat{S}$ is necessary.

Surrogate Dual Algorithm

Lower bound= 1074, Upper bound= 1449
Estimated duality gap = 25.89%

Upper bound achieved by Y = 1 1 1 1, i.e.,
opening plants 1 2 3 4

(Not guaranteed to be optimal!)

Surrogate Dual Algorithm

Matrix C÷⍉(⌽ρC)ρ(SD-F)

0.4198	0.0771	0.05997	0.318	0.2624	0.1414	0.2795	0
1.027	0.0629	0.07339	0	0.411	0.346	0.8209	0.736
0.3278	0.0669	0.05464	0.3312	0.2185	0.184	0.194	0.2609
0.8289	0.07612	0.0296	0.2442	0.259	0.3489	0.7358	0.6597

(Y[i]=1 if any column minimum,i.e., Lambda, is found in row # i of the matrix above)

Surrogate multipliers

j	1	2	3	4	5	6	7	8
Lambda[j]	0.3278	0.0629	0.0296	0	0.2185	0.1414	0.194	0

Theorem If $\mu_{ij} \geq 0$ and $\sum_{j=1}^{N} \mu_{ij} \leq F_i \ \forall i$

then $\sum_{j=1}^{N} \min_i \{C_{ij} + \mu_{ij}\}$ is a ***lower bound***

for the Simple Plant Location problem

Note: If $\mu_{ij} = \dfrac{F_i}{N} \ \forall i,j$ *, this is the lower bound provided by the LP relaxation of model #2! By appropriate choice of* μ_{ij} *, it may give us a better lower bound.*

Proof: SPL model #1 may be written

$$\Phi = \textbf{minimum} \sum_{i,j} C_{ij}X_{ij} + \sum_{i} \left(F_i - \sum_{j} \mu_{ij}\right)Y_i + \sum_{i,j} \mu_{ij}Y_i$$

$$\textbf{s.t.} \sum_{i} X_{ij} = 1, \quad X_{ij} \le Y_i, \; X_{ij} \ge 0, \; Y_i \in \{0,1\} \; \forall \mathbf{i,j}$$

$$\Rightarrow \Phi \ge \sum_{i,j} C_{ij}X_{ij} + \sum_{i,j} \mu_{ij}Y_i \ge \sum_{i,j} C_{ij}X_{ij} + \sum_{i,j} \mu_{ij}X_{ij} = \sum_{i,j} \left(C_{ij} + \mu_{ij}\right)X_{ij}$$

$$\Rightarrow \quad \textbf{minimum} \sum_{i,j} \left(C_{ij} + \mu_{ij}\right)X_{ij}$$

$$\textbf{s.t.} \sum_{i} X_{ij} = 1, \quad X_{ij} \le Y_i, \; X_{ij} \ge 0, \; Y_i \in \{0,1\} \; \forall \mathbf{i,j}$$

must give us a lower bound for SPL, namely

$$\sum_{j=1}^{N} \min_{i} \left\{C_{ij} + \mu_{ij}\right\}$$

The dual problem is, then, to choose the quantities μ_{ij} so as to obtain the ***greatest lower bound***, i.e.,

$$\text{Maximize} \sum_{j=1}^{N} \min_i \{C_{ij} + \mu_{ij}\}$$

$$\text{s.t.} \quad \sum_j \mu_{ij} \leq F_i \ \forall\ i$$

$$\mu_{ij} \geq 0 \ \forall\ i,j$$

$$\text{Maximize} \sum_{j=1}^{N} \min_i \{C_{ij} + \mu_{ij}\}$$

$$\text{s.t.} \quad \sum_j \mu_{ij} \le F_i \ \forall\, i$$

$$\mu_{ij} \ge 0 \ \forall\, i,j$$

The LP equivalent:

$$\text{Maximize} \sum_{j=1}^{N} Z_j$$

$$\text{s.t.} \quad Z_j \le C_{ij} + \mu_{ij} \ \forall\, i,j$$

$$\sum_j \mu_{ij} \le F_i \ \forall\, i$$

$$\mu_{ij} \ge 0 \ \forall\, i,j$$

The dual of this LP is, in fact, the ***LP relaxation*** *of SPL model #1!*

Bilde-Krarup-Erlenkotter [BKE] Algorithm

This algorithm is a dual ascent algorithm for computing good feasible solutions to the dual of the LP relaxation of Model #1.

At each iteration, exactly one μ_{ij} is adjusted to give an improvement in the lower bound. It terminates when no improvement can be obtained by adjusting a single multiplier.

Bilde-Krarup-Erlenkotter Dual Algorithm

Step 1: k←1 & Lambda← 294 60 28 0 196 132 174 0

Step 2a: ϵ= 98 0 0 0 0 0 0 0
Lambda[1]= 392
e= 0 0 98 0, LB= 982

Step 2a: ϵ= 98 0 0 0 0 0 0 0
Lambda[2]= 60
e= 0 0 98 0, LB= 982

Step 2a: ϵ= 98 0 21 0 0 0 0 0
Lambda[3]= 49
e= 0 0 98 21, LB= 1003

Step 2a: ϵ= 98 0 21 120 0 0 0 0
Lambda[4]= 120
e= 0 120 98 21, LB= 1123

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Step 2a: ∈= 98 0 21 120 49 0 0 0
Lambda[5]= 245
e= 0 120 147 21, LB= 1172

Step 2a: ∈= 98 0 21 120 49 33 0 0
Lambda[6]= 165
e= 33 120 147 21, LB= 1205

Step 2a: ∈= 98 0 21 120 49 33 30 0
Lambda[7]= 204
e= 33 120 177 21, LB= 1235

Step 2a: ∈= 98 0 21 120 49 33 30 107
Lambda[8]= 107
e= 140 120 177 21, LB= 1342

Step 3: do not terminate. Set k← 2

Step 2a: ∈= 0 0 21 120 49 33 30 107
Lambda[1]= 392
e= 140 120 177 21, LB= 1342
```

```
Step 2a: ϵ= 0 0 21 120 49 33 30 107
Lambda[2]= 60
e= 140 120 177 21, LB= 1342

Step 2a: ϵ= 0 0 0 120 49 33 30 107
Lambda[3]= 49
e= 140 120 177 21, LB= 1342

Step 2a: ϵ= 0 0 0 0 49 33 30 107
Lambda[4]= 120
e= 140 120 177 21, LB= 1342

Step 2a: ϵ= 0 0 0 0 0 33 30 107
Lambda[5]= 245
e= 140 120 177 21, LB= 1342

Step 2a: ϵ= 0 0 0 0 0 0 30 107
Lambda[6]= 165
e= 140 120 177 21, LB= 1342
```

```
Step 2a: ∈= 0 0 0 0 0 0 0 107
Lambda[7]= 204
e= 140 120 177 21, LB= 1342

Step 2a: ∈= 0 0 0 0 0 0 0 0
Lambda[8]= 107
e= 140 120 177 21, LB= 1342


Lower bound= 1342, Upper bound= 1342
Duality gap = 0%
No Duality Gap!

Upper bound achieved by Y = 1 1 1 0,
        i.e., opening plants 1 2 3
```

Lagrange multipliers

j	1	2	3	4	5	6	7	8
Lambda[j]	392	60	49	120	245	165	204	107

Summary of Results for Example Problem

		gap
Optimal Solution of SPL =	1342	—
LP Relaxation of Model #1 =	1342	0%
Surrogate Relaxation of Model #3 =	1074	20%
LP Relaxation of Model #2 =	1031.38	23%