SEPARABLE PROGRAMMING: REFINING THE GRID

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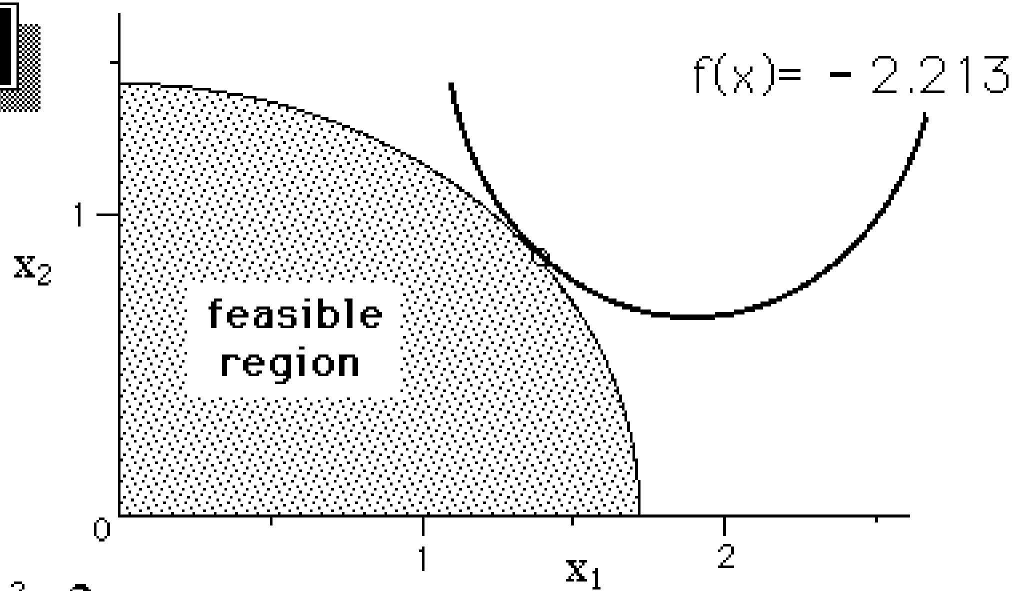
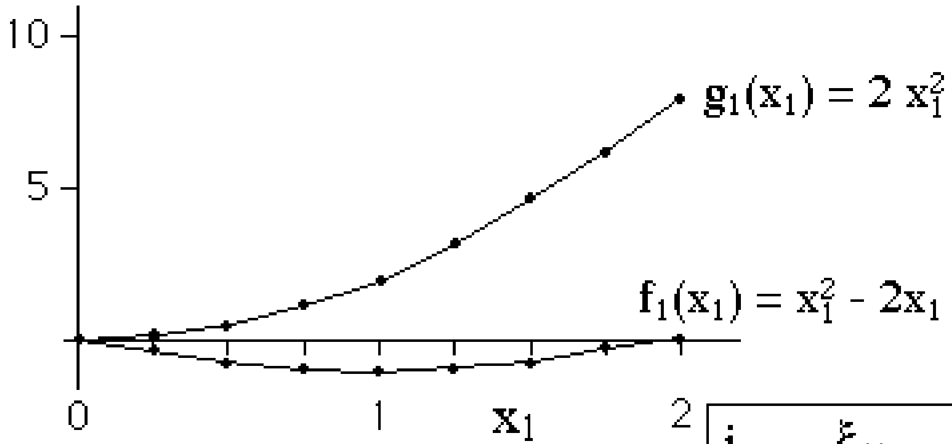
EXAMPLE



Minimize $x_1^2 - 2x_1 - x_2$

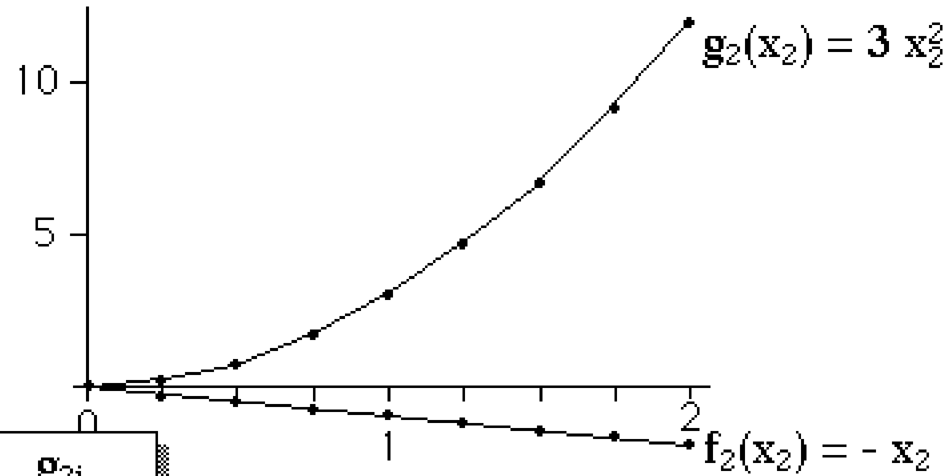
subject to $2\,x_1^2 + 3\,x_2^2 \leq 6$

$x_1 \geq 0,\ x_2 \geq 0$



Having determined that the constraints imply that $0 \leq x_1 \leq 2$ *, we select 9 grid points, in this case evenly distributed.*

i	ξ_{1i}	f_{1i}	g_{1i}
0	0	0	0
1	0.25	-0.4375	0.125
2	0.5	-0.75	0.5
3	0.75	-0.9375	1.125
4	1.0	-1.0	2.0
5	1.25	-0.9375	3.125
6	1.5	-0.75	4.5
7	1.75	-0.4375	6.125
8	2.0	0	8.0



i	ξ_{2i}	f_{2i}	g_{2i}
0	0	0	0
1	0.25	−0.25	0.1875
2	0.5	−0.5	0.75
3	0.75	−0.75	1.6875
4	1.0	−1.0	3.0
5	1.25	−1.25	4.6875
6	1.5	−1.5	6.75
7	1.75	−1.75	9.1875
8	2.0	−2.0	12.0

Likewise, we determine that feasibility requires that $0 \leq x_2 \leq 2$ *, and select 9 grid points for* x_2.

The piecewise-linear approximation has the LP formulation:

$$\text{Minimize} \sum_{i=1}^{2} \sum_{j=0}^{8} \lambda_{ij} f_{ij}$$

$$\text{subject to} \sum_{i=1}^{2} \sum_{j=0}^{8} \lambda_{ij} g_{ij} \leq 6$$

$$\sum_{j=0}^{8} \lambda_{ij} = 1, \ \forall\, i$$

$$\lambda_{ij} \geq 0, \ \forall\, i \ \& \ j$$

That is, the LP problem:

$$\text{Min } -0.4375\lambda_{11}-0.75\lambda_{12}-0.9375\lambda_{13}-\lambda_{14}-0.9375\lambda_{15}-0.75\lambda_{16}-0.4375\lambda_{17}$$
$$-0.25\lambda_{21}-0.5\lambda_{22}-0.75\lambda_{23}-\lambda_{24}-1.25\lambda_{25}-1.5\lambda_{26}-1.75\lambda_{27}-2\lambda_{28}$$

subject to

$$1.25\lambda_{11} + 0.5\lambda_{12} + 1.125\lambda_{13} + 2\lambda_{14}$$
$$+ 3.125\lambda_{15} + 4.5\lambda_{16} + 6.125\lambda_{17} + 8\lambda_{18}$$
$$+ 0.1875\lambda_{21} + 0.75\lambda_{22} + 1.6875\lambda_{23} + 3\lambda_{24}$$
$$+ 4.6875\lambda_{25} + 6.75\lambda_{26} + 9.1875\lambda_{27} + 12\lambda_{28} \leq 6$$

"convexity" constraints

$$\begin{cases} \lambda_{11}+\lambda_{12}+\lambda_{13}+\lambda_{14}+\lambda_{15}+\lambda_{16}+\lambda_{17}+\lambda_{18} = 1 \\ \lambda_{21}+\lambda_{22}+\lambda_{23}+\lambda_{24}+\lambda_{25}+\lambda_{26}+\lambda_{27}+\lambda_{28} = 1 \end{cases}$$

$$\lambda_{ij} \geq 0, \ \forall \text{ i \& j}$$

...at most TWO λ_i's will be positive, and these will be weights of adjacent grid points!

$\lambda_{10} = 0$
$\lambda_{11} = 0$
$\lambda_{12} = 0$
$\lambda_{13} = 1$
$\lambda_{14} = 0$
$\lambda_{15} = 0$
$\lambda_{16} = 0$
$\lambda_{17} = 0$
$\lambda_{18} = 0$

$\lambda_{20} = 0$
$\lambda_{21} = 0$
$\lambda_{22} = 0$
$\lambda_{23} = 0$
$\lambda_{24} = 0$
$\lambda_{25} = 0.9090909$
$\lambda_{26} = 0.0909090$
$\lambda_{27} = 0$
$\lambda_{28} = 0$

The LP solution displays the property we expect when the problem is convex!

LP Solution

LP Solution

$$X_1 = 0.75\lambda_{13}$$
$$= (0.75)(1) = 0.75$$
$$X_2 = 1.25\lambda_{25} + 1.5\lambda_{26}$$
$$= (1.25)\,(0.9090909) + (1.5)\,(0.0909090)$$
$$= 1.2727$$
$$\text{LP objective} = 2.21023$$

The solution obtained from the piecewise-linear approximation is reasonably close to the "true" optimum:

$$X_1^* = 0.7906$$
$$X_2^* = 1.258$$
$$f(X^*) = 2.213$$

Piecewise-Linear Approximation of Convex Nonlinear Separable Programs

$$\text{Minimize} \quad \sum_{j=1}^{n} f_j(x_j)$$

$$\text{subject to} \quad \sum_{j=1}^{n} g_{ij}(x_j) \le b_i,\ i=1,\cdots m$$

$$x_j \ge 0,\ j=1,\cdots n$$

Given a set of p_j grid points $\{\gamma_{jk}\}_{k=1}^{p_j}$ for X_j, and assuming that, for each j, f_j & g_{ij} are convex functions, we obtain a piecewise-linear approximation:

$$\text{Minimize } \sum_{j=1}^{n} \sum_{k=1}^{p_j} f_j(\gamma_{jk})\,\lambda_{jk}$$

$$\text{subject to } \sum_{j=1}^{n} \sum_{k=1}^{p_j} g_{ij}(\gamma_{jk})\,\lambda_{jk} \le b_i,\ i=1,\cdots m$$

$$\sum_{k=1}^{p_j} \lambda_{jk} = 1,\ j=1,\cdots n$$

$$\lambda_{ik} \ge 0,\ j=1,\cdots n$$

The "finer" the mesh of the grid, i.e., the nearer the grid points, the more accurate is the piecewise-linear approximation, generally....
But the greater the computational burden!

What is needed is a "fine" mesh only in the vicinity of the optimal solution, with a coarse mesh elsewhere. The "grid refinement" method to be introduced next will iteratively select additional grid points to improve the approximation in the vicinity of the optimum!

Refining the Grid

Minimize $x_1^2 - 2x_1 - x_2$

subject to $2x_1^2 + 3x_2^2 \leq 6$

$x_1 \geq 0,\ x_2 \geq 0$

EXAMPLE

Instead of initially using 9 grid points for each variable, as before, we will use an initial "rough" grid, { 0, 1, 2 } for both x_1 & x_2

x_1	$x_1^2 - 2x_1$	$2x_1^2$
0	0	0
1	-1	2
2	0	8

x_2	$-x_2$	$3x_2^2$
0	0	0
1	−1	3
2	−2	12

Piecewise-Linear Approximation

Minimize $-\lambda_{12} - \lambda_{22} - 2\lambda_{23}$

subject to

$$2\lambda_{12} + 8\lambda_{13} + 3\lambda_{22} + 12\lambda_{23} \leq 6$$

$$\lambda_{11} + \lambda_{12} + \lambda_{13} = 1$$

$$\lambda_{21} + \lambda_{22} + \lambda_{23} = 1$$

$$\lambda_{jk} \geq 0 \text{ , } \forall \text{ j \& k}$$

LP Solution:

$$z = -\,^{19}/_{9}$$

$$\lambda_{12}^{*} = 1$$

$$\lambda_{22}^{*} = {}^{8}/_{9},\ \lambda_{23}^{*} = {}^{1}/_{9}$$

$$z = -\,{}^{19}\!/_{9}$$

$$\lambda_{12}^{*} = 1 \quad \Rightarrow x_1^{*} = (0)(0) + (1)(1) + (2)(0) = 1$$

$$\lambda_{22}^{*} = {}^{8}\!/_{9},\ \lambda_{23}^{*} = {}^{1}\!/_{9} \quad \Rightarrow x_2^{*} = (0)(0) + (1)({}^{8}\!/_{9}) + (2)({}^{1}\!/_{9}) = {}^{10}\!/_{9}$$

Optimal Simplex Multipliers (dual variables):

$$\pi = \left[-\,{}^{1}\!/_{9}\ ,\ -\,{}^{7}\!/_{9}\ ,\ -\,{}^{2}\!/_{3}\right]$$

How can we "refine the grid, i.e., add additional grid points, so as to get a better approximation and a better solution?

If γ_1 were a new grid point for x_1, then we would generate a new column for the tableau:

$$\begin{bmatrix} 2\gamma_1^2 \\ 1 \\ 0 \end{bmatrix} \quad \text{with cost coefficient} \quad \gamma_1^2 - 2\gamma_1$$

and **reduced cost**

$$\underbrace{\left(\gamma_1^2 - 2\gamma_1\right)}_{\text{cost of variable}} - \underbrace{\left[-1/9, -7/9, -2/3\right]}_{\text{simplex multipliers}} \underbrace{\begin{bmatrix} 2\gamma_1^2 \\ 1 \\ 0 \end{bmatrix}}_{\text{column of coefficients}}$$

reduced cost

$$\left(\gamma_1^2 - 2\gamma_1\right) - \left[-1/9, -7/9, -2/3\right]\begin{bmatrix} 2\gamma_1^2 \\ 1 \\ 0 \end{bmatrix}$$

cost of variable — simplex multipliers — column of coefficients

$$= \gamma_1^2 - 2\gamma_1 + \left(\frac{1}{9}\right)\left(2\,\gamma_1^2\right) + \left(\frac{7}{9}\right)(1) + \left(\frac{2}{3}\right)(0)$$

$$= \frac{11}{9}\,\gamma_1^2 - 2\,\gamma_1 + \frac{7}{9}$$

Given a choice of grid points to choose from, let's select that grid point whose column, when added to the LP tableau, has the smallest (i.e., "most negative") reduced cost.

Note that this rule does not necessarily give us the grid point which will yield the most improvement in the approximation or the objective function.

To identify this grid point, we will minimize the reduced cost, $\frac{11}{9}\gamma_1^2 - 2\gamma_1 + \frac{7}{9}$, which is a function of γ_1

Differentiating the reduced cost function

$$\frac{11}{9}\gamma_1^2 - 2\gamma_1 + \frac{7}{9}$$

and equating the derivative to zero yields (in this example) a linear equation which is easily solved for the grid point γ_1:

$$2\left(\frac{11}{9}\right)\gamma_1 - 2 = 0 \quad \Rightarrow \gamma_1 = \frac{9}{11}$$

with reduced cost $-0.0404 < 0$

The column which we therefore generate for the LP tableau, corresponding to this new grid point, is

$$\begin{bmatrix} 2\gamma_1^2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.3388 \\ 1 \\ 0 \end{bmatrix}$$

with objective coefficient

$$\gamma_1^2 - 2\gamma_1 = -0.96694$$

Likewise, selection of a new grid point for x_2 is done by choosing γ_2 in order to minimize the reduced cost of the generated column

$$-\gamma_2 - \left[-\tfrac{1}{9}, -\tfrac{7}{9}, -\tfrac{2}{3}\right]\begin{bmatrix} 3\gamma_2^2 \\ 0 \\ 1 \end{bmatrix} = \tfrac{1}{3}\gamma_2^2 - \gamma_2 + \tfrac{1}{3}$$

whose derivative, $\tfrac{2}{3}\gamma_2 - 1$, is zero at $\gamma_2 = \tfrac{3}{2}$

Thus, we refine the grids:

$$\{0, 9/11, 1, 2\} \quad \text{for } x_1$$
$$\{0, 1, 3/2, 2\} \quad \text{for } x_2$$

generate the new columns for the LP tableau, and re-optimize the LP:

$$\text{Minimize } -\lambda_{12} - \lambda_{22} - 2\lambda_{23} - 0.96694\lambda_{14} - 1.5\lambda_{24}$$
$$2\lambda_{12} + 8\lambda_{13} + 3\lambda_{22} + 12\lambda_{23} + 1.3388\lambda_{14} + 6.75\lambda_{24} \leq 6$$
$$\lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{14} = 1$$
$$\lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{24} = 1$$
$$\lambda_{jk} \geq 0, \ \forall\, j \ \& \ k$$

New LP optimum

$z = -2.1884$

$\lambda_{14} = 1, \lambda_{11} = \lambda_{12} = \lambda_{13} = 0$

$\lambda_{22} = 0.5570, \lambda_{24} = 0.4430, \lambda_{21} = \lambda_{23} = 0$

$\Rightarrow$

$x_1 = 0.8182$

$x_2 = 1.2215$

with Simplex multiplier vector

$$\pi = [\, -0.1333\ ,\ -0.7884\ ,\ -0.6\,]$$

Let's further refine the grid

- Reduced cost for grid point γ_1's column is

$$(\gamma_1^2 - 2\gamma_1) + 0.1333\,(2\gamma_1^2) + 0.7884$$

which is minimized (with value - 0.0010) at $\gamma_1 = 0.7895$

- Reduced cost for grid point γ_2's column is

$$-\gamma_2 + 0.1333\,(3\gamma_2^2) + 0.6$$

which is minimized (with value -0.025) at $\gamma_2 = 1.25$

Next LP

Minimize $-\lambda_{12} - \lambda_{22} - 2\lambda_{23} - 0.96694\lambda_{14} - 1.5\lambda_{24} - 0.95569\lambda_{15} - 1.25\lambda_{25}$

subject to

$$2\lambda_{12} + 8\lambda_{13} + 3\lambda_{22} + 12\lambda_{23} + 1.3388\lambda_{14} + 6.75\lambda_{24} + 1.2466\,\lambda_{15} + 4.6875\lambda_{25} \le 6$$

$$\lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{14} + \lambda_{15} = 1$$

$$\lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{24} + \lambda_{25} = 1$$

$$\lambda_{jk} \ge 0\ ,\ \forall\ j\ \&\ k$$

which has optimum -2.2137 at

$\lambda_{14} = 0.714751, \lambda_{15} = 0.285249$
$\lambda_{25} = 1$ $\Rightarrow$ $x_1 = 0.8100$, $x_2 = 1.25$

General Scheme

Suppose that the simplex multipliers are

$$\pi = [\, \pi_1 , \pi_2 , \cdots \pi_m \mid \pi_{m+1} , \pi_{m+2} , \cdots \pi_{m+n}]$$

$$\underbrace{\pi_1 , \pi_2 , \cdots \pi_m}_{\text{regular constraints}} \quad \underbrace{\pi_{m+1} , \pi_{m+2} , \cdots \pi_{m+n}}_{\text{convexity constraints}}$$

These simplex multipliers are used by the revised simplex method to compute the reduced cost of a nonbasic variable.

Corresponding to a new grid point γ_j for x_j is the LP column

$$\begin{array}{ll} \text{regular constraints} & \begin{bmatrix} g_{1j}(\gamma_j) \\ g_{2j}(\gamma_j) \\ \vdots \\ g_{mj}(\gamma_j) \\ \hdashline 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \\ \text{convexity constraints} & \end{array}$$

with objective coefficient

$$f_j(\gamma_j)$$

and reduced cost function

$$f_j(\gamma_j) - \sum_{i=1}^{m} \pi_i g_{ij}(\gamma_j) - \pi_{m+j}$$

For each j=1, 2, ... n:

- find the grid point γ_j which minimizes the reduced cost function

$$f_j(\gamma_j) - \sum_{i=1}^{m} \pi_i g_{ij}(\gamma_j) - \pi_{m+j}$$

- if the value of the reduced cost function exceeds some tolerance $\varepsilon > 0$ in absolute value, generate the LP column and add to the tableau

If no new column was added to the LP tableau, then terminate.

Otherwise, re-optimize the LP, and repeat the procedure.

Note that in our example, we were able to minimize the reduced cost function analytically; more generally, it is necessary to use a one-dimensional search technique (e.g., golden section search, fibonacci search, quadratic interpolation, etc.)