Stochastic Transportation Problem

non-simple recourse
normally-distributed demand

Stochastic Decomposition

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DATA
Stochastic Transportation

First-stage data:

A,B=

```
1 1 1 0 0 0 0 0 0 = 9
0 0 0 1 1 1 0 0 0 = 3
0 0 0 0 0 0 1 1 1 = 8
```

i	variable	cost
1	X11	0
2	X12	2
3	X13	3
4	X21	2
5	X22	0
6	X23	2
7	X31	3
8	X32	2
9	X33	0

Objective: Minimize

Second-stage data

Costs:

i	variable	q	
1	Y12	6	
2	Y13	10	
3	Y21	6	
4	Y23	15	
5	Y31	12	
6	Y32	15	
7	EX1	-4	<-***excess***
8	EX2	-4	***supply***
9	EX3	-2	
10	SH1	15	<-***shortage***
11	SH2	20	***of supply***
12	SH3	30	

Technology matrix T
(coefficients of X in 2nd stage) =

```
1 0 0 1 0 0 1 0 0
0 1 0 0 1 0 0 1 0
0 0 1 0 0 1 0 0 1
```

Technology matrix W
(coefficients of Y in 2nd stage) =

```
     -1 -1  1  0  1  0 -1  0  0 1 0 0
      1  0 -1 -1  0  1  0 -1  0 0 1 0
      0  1  0  1 -1 -1  0  0 -1 0 0 1
```

(Only the right-hand-side vector is random!)
Right-hand-sides in second stage =

i	mean	std dev
1	6	2
2	7	2
3	7	3

random demands

Certainty-Equivalent Tableau

Using expected values for right-hand-sides

b	z	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9	10	11	12
0	1	0	2	3	2	0	2	3	2	0	6	10	6	15	12	15	-4	-4	-2	15	20	30
9	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
6	0	1	0	0	1	0	0	1	0	0	-1	-1	1	0	1	0	-1	0	0	1	0	0
7	0	0	1	0	0	1	0	0	1	0	1	0	-1	-1	0	1	0	-1	0	0	1	0
7	0	0	0	1	0	0	1	0	0	1	0	1	0	1	-1	-1	0	0	-1	0	0	1

Optimal Solution

Total objective function: 8

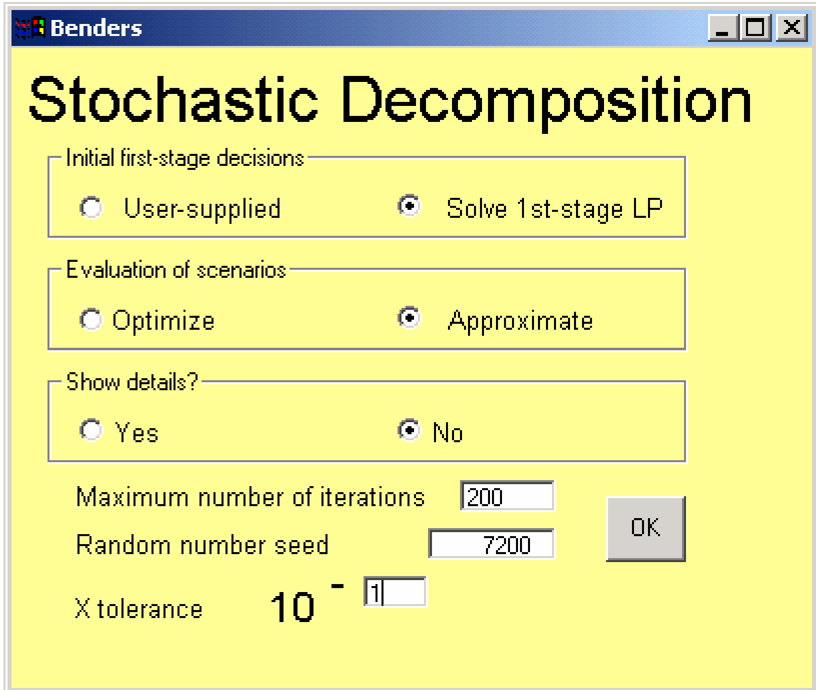
Stage One: nonzero variables:

i	variable	value
1	X11	6
2	X12	3
5	X22	3
8	X32	1
9	X33	7

Second Stage: nonzero variables

i	variable	value

--none—



We use the "Stochastic Decomposition" method of Higle & Sen, which approximates Benders' decomposition.

Iteration #1

Trial X for primal subproblems (#1) is

i	Variable	Value	
1	X11	6	*(using solution*
2	X12	3	*of the*
5	X22	3	*certainty-*
8	X32	1	*equivalent*
9	X33	7	*problem)*

Solve subproblem with new trial x (#1) :

RHS = 6.96191 10.2626 7.11435 *(1st scenario)*

Second-stage cost: 82.539

Optimal dual vector: 15 20 25 *(1st dual sol'n λ)*

Newly-generated optimality cut at iteration 1

s	i	beta	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
1	1	487.539	-15	-20	-25	-15	-20	-25	-15	-20	-25

s is scenario #, i is dual solution #, beta is constant

Primal subproblems summary

First stage cost: 8
Second stage costs:

s	Lambda#	cost
1	1	82.539

Average second stage cost: 82.539
Total: **90.539**

Solution of Master Problem

X= 6 3 0 0 3 0 0 1 7
First-stage cost: 33
Estimated second-stage cost Q(X) = -12.461
Total (estimated) expected value: 20.539

Iteration #2

Trial X for primal subproblems (#2) is

i	Variable	Value
3	X13	9
6	X23	3
9	X33	8

Solve subproblem with new trial x (#2) :
Primal Subproblem Result:
RHS = 6.70624 7.76354 7.56864 *(2nd scenario)*
Second-stage cost: 203.043
Optimal dual vector: 15 18 3 (2nd dual sol'n λ)

Solve subproblem with incumbent solution (#1) :
Primal Subproblem Result:
RHS = 6.70624 7.76354 7.56864
Second-stage cost: 40.0802
Optimal dual vector: 15 20 25 (1st λ again!)

Newly-generated optimality cut at iteration 2

s	i	beta	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
1	1	487.539	-15	-20	-25	-15	-20	-25	-15	-20	-25
2	1	445.08	-15	-20	-25	-15	-20	-25	-15	-20	-25

s is scenario #, i is dual solution #, beta is constant

Aggregate cut:

beta	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
466.31	-15	-20	-25	-15	-20	-25	-15	-20	-25

Primal subproblems summary

First stage cost: 33

Second stage costs:

s	Lambda#	cost
1	1	-12.4610
2	1	40.0802

Average second stage cost: 13.8096

Total: 46.8096

Solution of Master Problem

X= 0 0 9 0 0 3 0 0 8

First-stage cost: 33

Estimated second-stage cost Q(X) = -12.461

Total (estimated) expected value: 20.539

Iteration #3

Trial X for primal subproblems (#2) is

i	Variable	Value
3	X13	9
6	X23	3
9	X33	8

Solve subproblem with new trial x (#2) :
Primal Subproblem Result:

RHS = 5.48475 5.35459 13.8181 *(3rd scenario)*

Second-stage cost: 160.108

Optimal dual vector: 15 18 3 *(2nd λ again!)*

Newly-generated optimality cut at iteration 3

s	i	beta	[1]	[2]	3]	[4]	[5]	6]	[7]	[8]	9]
1	2	310.498	-15	-18	-3	-15	-18	-3	-15	-18	-3
2	2	263.043	-15	-18	-3	-15	-18	-3	-15	-18	-3
3	2	220.108	-15	-18	-3	-15	-18	-3	-15	-18	-3

s is scenario #, i is dual solution #, beta is constant

Aggregate cut:

beta	[1]	[2]	3]	[4]	[5]	6]	[7]	[8]	9]
264.55	-15	-18	-3	-15	-18	-3	-15	-18	-3

Primal subproblems summary

First stage cost: 33
Second stage costs:

s	Lambda#	cost
1	2	203.043
2	2	250.498
3	2	160.108

Average second stage cost: 204.55
Total: 237.55

Solution of Master Problem

X= 0 0 9 0 0 3 0 0 8
First-stage cost: 18.3896
Estimated second-stage cost Q(X) = 30.394
Total (estimated) expected value: 48.7836

...etc.

Summary of 200 iterations

```
Stochastic Decomposition

Random number seed used in computation: 7200
Lower bound used in updating old cuts: 0
Method: Subproblems solved  approximately
Tolerance for distinguishing first-stage solutions X:
1.0E-1

# iterations (= # right-hand-sides sampled): 200
# second-stage problems solved: 397

# first-stage solutions generated: 79
Best solution found is #68 with estimated cost 46.3373
23 second-stage problems were solved using this X

# second-stage dual solutions generated: 16
```

Evaluation of trial solution # 68

i	variable	X[i]
1	X11	6.566589
2	X12	2.433411
5	X22	3.000000
8	X32	0.338095
9	X33	7.661905

(Using optimality cuts as approximation of expected second-stage cost.)

First stage objective:	5.54
Expected second stage objective:	41.48
Total:	47.03

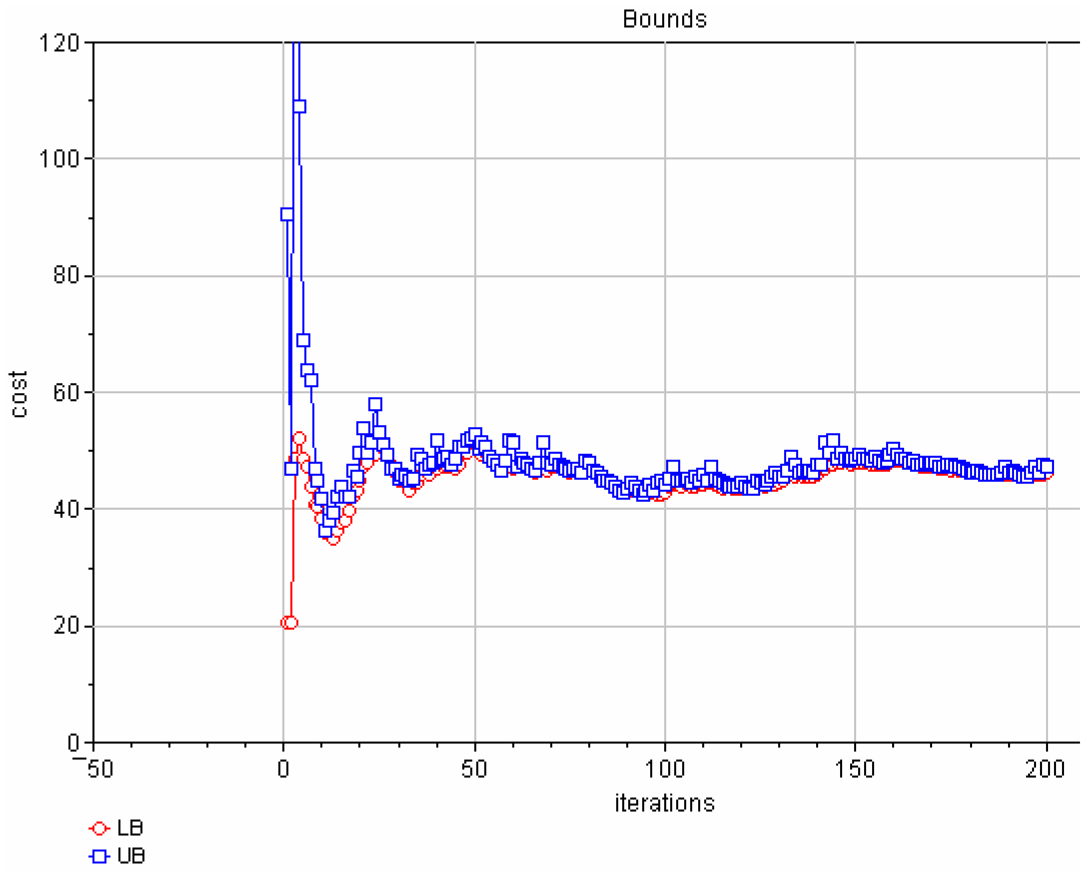
(Using expected second-stage costs approximated by restriction to 16 recorded dual solutions.)

First stage objective:	5.54
Expected second stage objective:	40.07
Total:	45.61

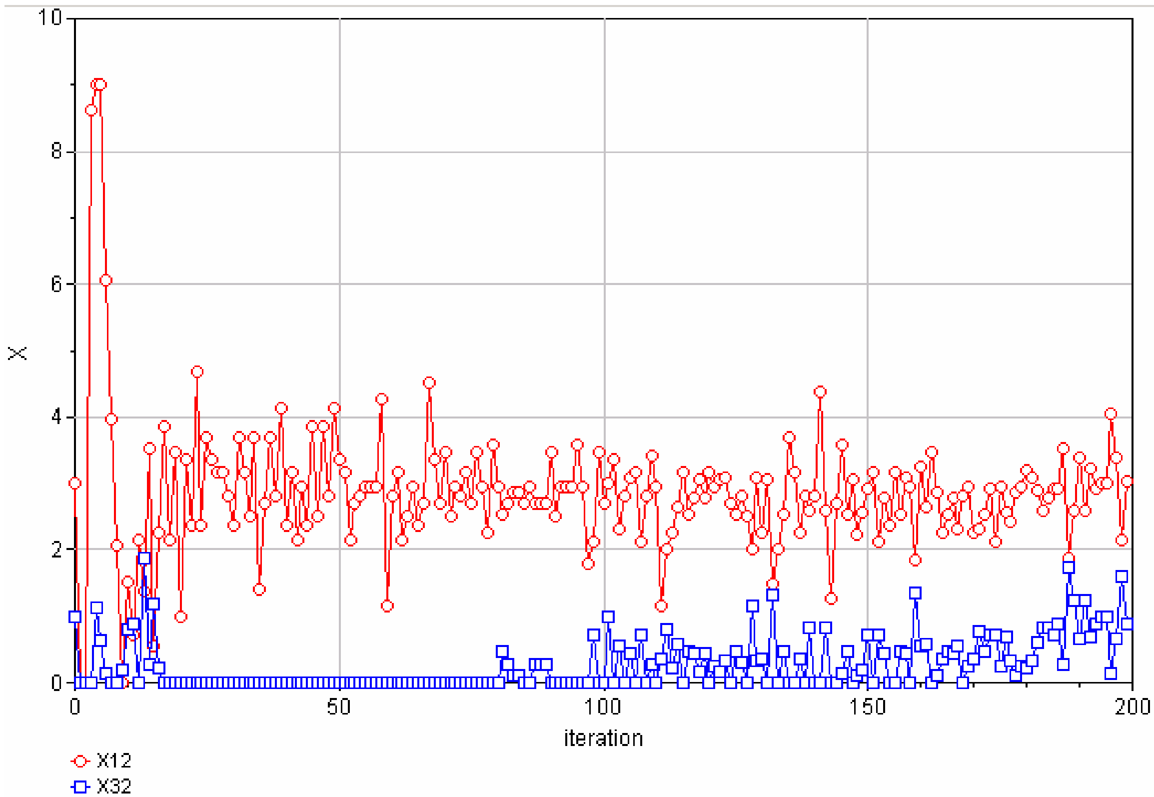
(Using 23 evaluations of second-stage costs.)

First stage objective:	5.54
Expected second stage objective:	33.85
Total:	39.39

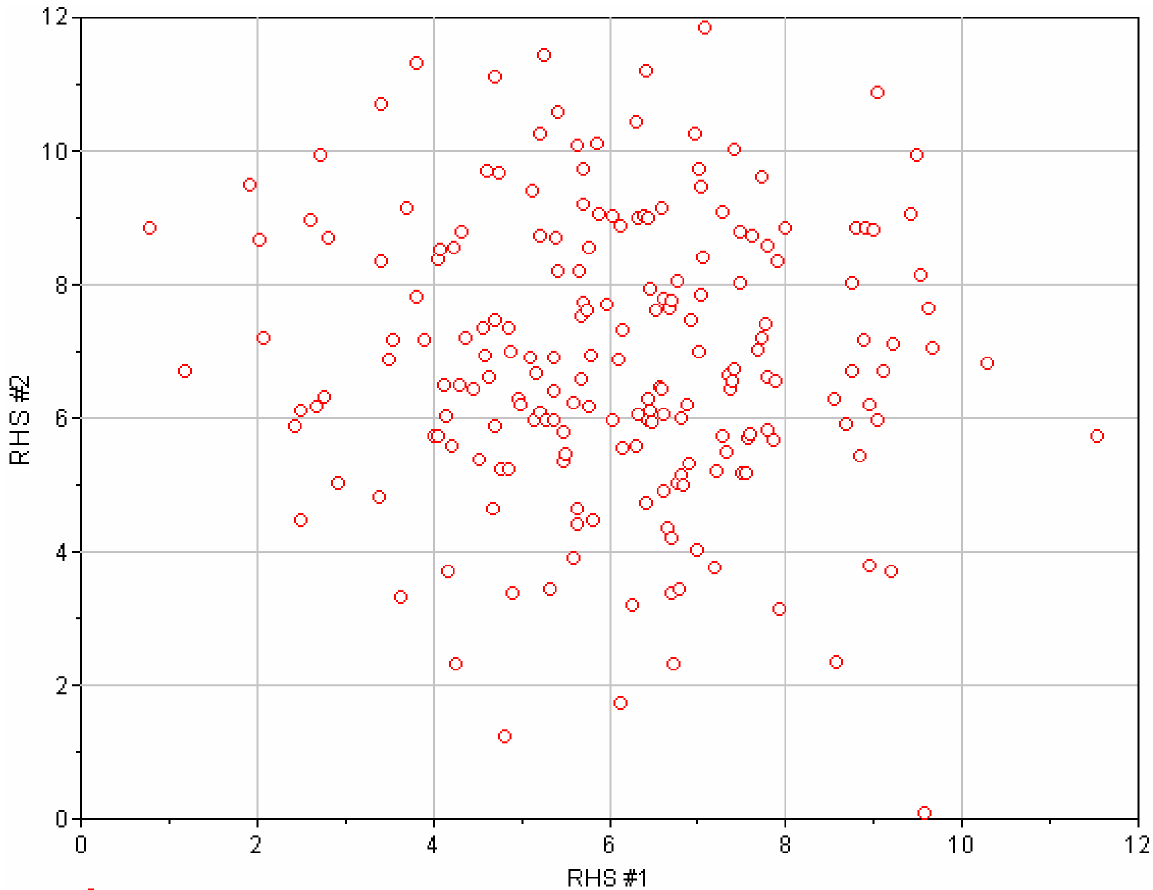
Plot of upper & lower "bounds"



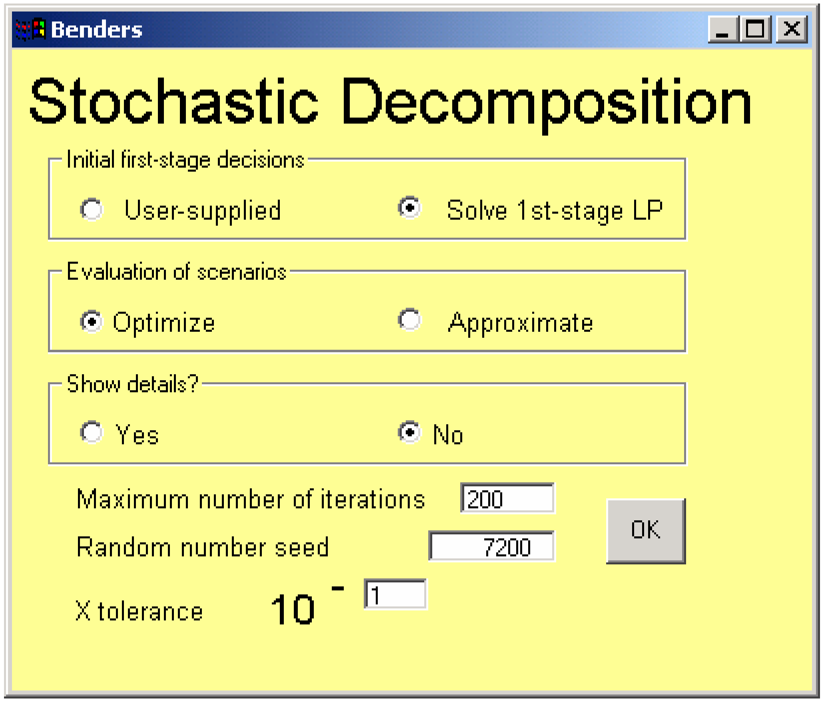
Plot of variables X12 & X32 vs iteration



Plot of the randomly-generated right-hand-sides of constraints 1&2



Suppose we were to evaluate the 1st stage solutions by solving an LP:



(Same random number seed is used so that same scenarios will be generated.)

Summary

```
Stochastic Decomposition

Random number seed used in computation: 7200
Lower bound used in updating old cuts: 0
Method: Subproblems solved  exactly
Tolerance for distinguishing first-stage solutions:1.0E-1

# iterations (= # right-hand-sides sampled): 200
# second-stage problems solved: 5330

# first-stage solutions generated: 73
Best solution found is #1 with estimated cost 47.4702
200 second-stage problems were solved using this X

# second-stage dual solutions generated: 18
```

Evaluation of trial solution # 1

i	variable	X[i]
1	X11	6
2	X12	3
5	X22	3
8	X32	1
9	X33	7

(Using optimality cuts as approximation of expected second-stage cost.)

First stage objective:	8.00
Expected second stage objective:	82.54
Total:	90.54

(Using expected second-stage costs approximated by restriction to 18 recorded dual solutions.)

First stage objective:	8.00
Expected second stage objective:	45.67
Total:	53.67

(Using 200 evaluations of second-stage costs.)

First stage objective:	8.00
Expected second stage objective:	39.47
Total:	47.47

We'll try 500 iterations with a different random number seed:

<pre>
Stochastic Decomposition

Random number seed used in computation: 7179
Lower bound used in updating old cuts: 0
Method: Subproblems solved approximately
Tolerance for distinguishing first-stage solutions: 1.0E-1

iterations (= # right-hand-sides sampled): 500
second-stage problems solved: 994

first-stage solutions generated: 93
Best solution found is #92 with estimated cost 50.5342
309 second-stage problems were solved using this X

second-stage dual solutions generated: 18
</pre>

Evaluation of trial solution # 92

i	variable	X[i]
1	X11	6.51181
2	X12	2.48819
5	X22	3.00000
9	X33	8.00000

Using optimality cuts as approximation of expected second-stage cost:

First stage objective:	4.98
Expected second stage objective:	49.42
Total:	54.40

Using expected second-stage costs approximated by restriction to 18 recorded dual solutions:

First stage objective:	4.98
Expected second stage objective:	44.26
Total:	49.24

Using 309 evaluations of second-stage costs:

First stage objective:	4.98
Expected second stage objective:	43.91
Total:	48.88