# Relaxing the Exponential Assumption

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In queues which are modeled as birth-death processes,

both the times between arrivals
and the service times

must have exponential distributions.

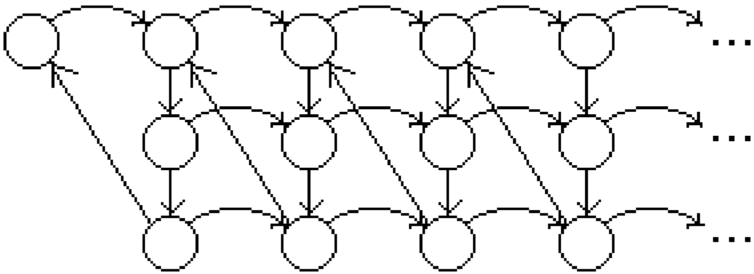
Exponential distributions have coefficient of variation equal to 1, i.e.,

$$\frac{\sqrt{var[T]}}{E[T]} = 1$$

If, in an application, inter-arrival &/or service times are either more or less *regular*, what can be done?

We have seen that an $M/E_k/1$ queue, for which service times have Erlang-k distribution, can be modeled as a (continuous-time) Markov chain.

That is, the service consists of k phases, each with exponentially-distributed service time.



If service time T has Erlang-k dist'n with mean $1/k\mu$,

then

$$T = \sum_{i=1}^{k} Y_i \,,\quad E[T] = \sum_{i=1}^{k} E[Y_i] = kE[Y_i] = 1/\mu$$

$$Var[T] = \sum_{i=1}^{k} Var[Y_i] = k \times Var[Y_i] = k/(k\mu)^2 = 1/k\mu^2$$

$$\Rightarrow \text{Coefficient of variation} = 1/\sqrt{k}$$

*The coefficient of variation may be made as small as we like (but >0) by increasing k.*

An Erlang-k random variable is a *convolution* of k random variables with exponential dist'n, and is *more regular* than a random variable with exponential distribution.

To approximate distributions which are *less regular*, i.e., have c.v. > 1, we can use a *hyper-exponential* distribution.

$$P\{T \le t\} = F(t) = \beta\left[1 - e^{-\mu_1 t}\right] + (1-\beta)\left[1 - e^{-\mu_2 t}\right]$$

where $0 < \beta < 1$.

# Hyper-Exponential Distribution

A hyper-exponential dist'n is a *mixture* of exponential distributions, with service rate $\mu$ defined by

$$\frac{1}{\mu} = \frac{\beta}{\mu_1} + \frac{(1-\beta)}{\mu_2}$$

and has coefficient of variation $>1$ and can be made arbitrarily large.

# M/H/1 Queue

Arrivals are Poisson with rate $\lambda$ , and service time has hyper-exponential dist'n

$$F(t) = \beta\left[1-e^{-\mu_1 t}\right] + (1-\beta)\left[1-e^{-\mu_2 t}\right]$$

Equivalently, suppose 2 types of customers,

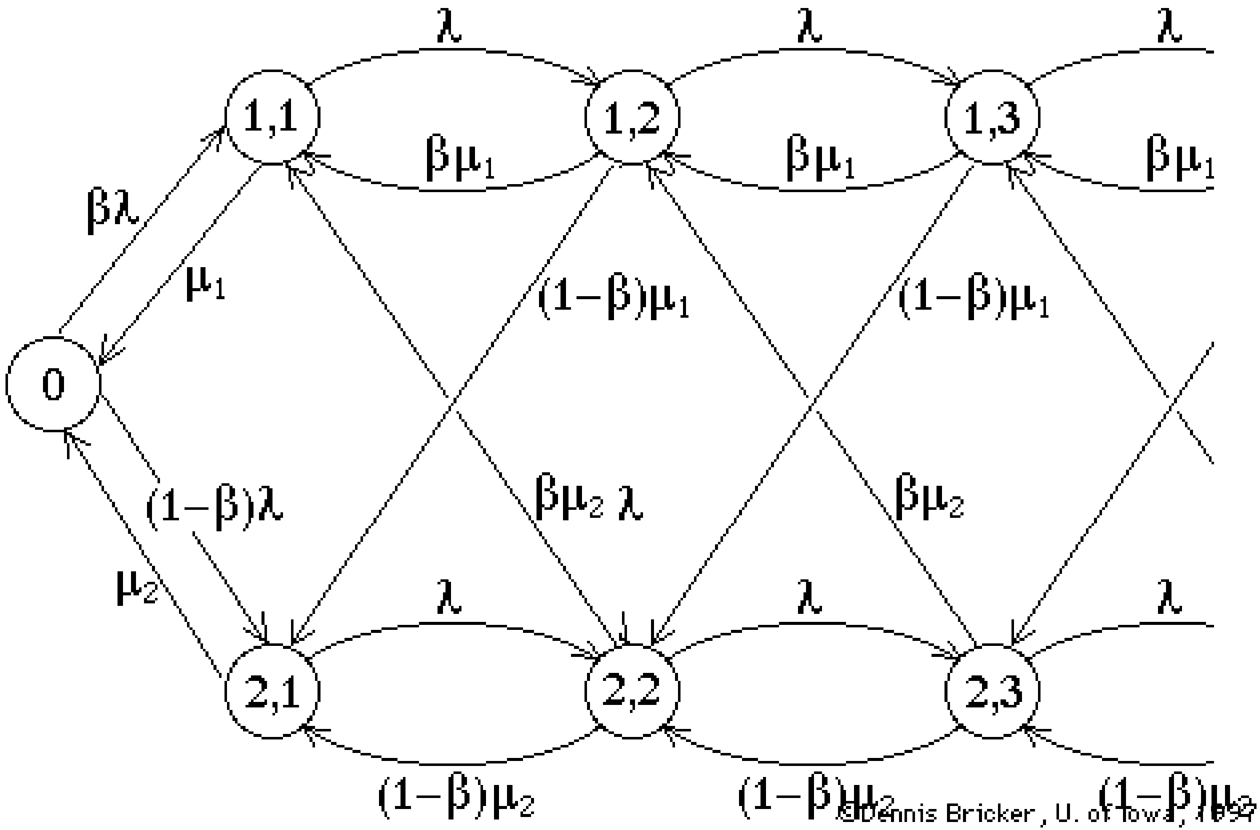
Type 1 with service time distn Exp($\mu_1$)

Type 2 with service time distn Exp($\mu_2$)

where $\beta$ = fraction of customers that are type 1

# States (i,j)

where
- i=type of service being provided
- j=# in system



λ
λ
λ
1,1
1,2
1,3
βμ1
βμ1
βμ1
βλ
μ1
(1−β)μ1
(1−β)μ1
0
(1−β)λ
βμ2 λ
βμ2
μ2
λ
λ
λ
2,1
2,2
2,3
(1−β)μ2
(1−β)μ2
(1−β)μ2

## Phase-type Distribution

A distribution function H is phase-type with k phases

if it is $(1-\beta_1)F_1 + (1-\beta_2)\beta_1 F_1 \oplus F_2 + \cdots$

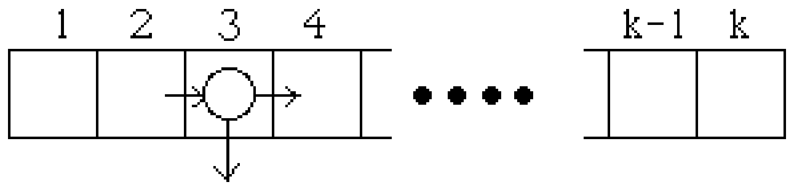
$$+\beta_1\beta_2 \cdots \beta_{k-1} F_1 \oplus F_2 \oplus \cdots \oplus F_k$$

where $F_i$ is exponential dist'n with rate $\mu_i$

$F_1 \oplus F_2$ is the convolution of $F_1$ and $F_2$,

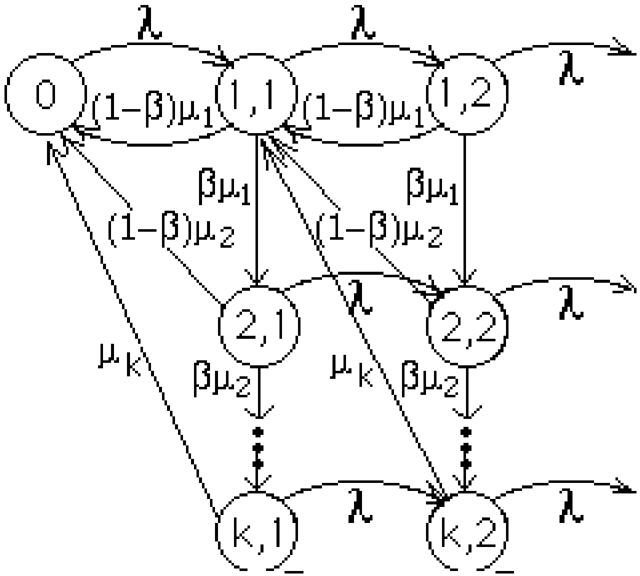
$0 < \beta_i < 1$ for $i=1,2,\ldots k-1$, $\beta_k = 0$

# Phase-type Distribution




Think of the service facility as having k stages, with a service time in stage j having exponential dist'n (rate $\mu_j$),

and upon completion of stage j, the service is complete with probability $(1-\beta_j)$ or customer enters stage j+1.




## States (i,j)

i=phase of service in progress

j= # of customers in the system

# Phase-type Distribution

The Erlang & Hyper-exponential dist'ns are both phase-type distributions.

Any arbitrary distribution can be approximated as closely as desired by a phase-type dist'n.

**example** Suppose that service time T is discrete, with $P\{T=a\} = 1-\beta$ and $P\{T=b\} = \beta$, for $\beta \in (0,1)$.

That is, $T = a + (b-a)\,I$

where $P\{I=1\}=\beta$, $P\{I=0\}= 1-\beta$.

Consider the distribution $(1-\beta)\,E_{k'} + \beta\,E_{k'} \oplus E_{k''}$

where $E_{k'}$ is Erlang-k' with phase rate $k'/a$

$E_{k''}$ is Erlang-k" with phase rate $k''/(b-a)$

As $k' \to \infty$ & $k'' \to \infty$, this dist'n converges to that of T.

Using phase-type distributions, we can, in principle, approximate any queue as a continuous-time Markov chain....

In practice, the state space of this Markov chain may be large &/or complex, and the balance equations intractable.

As the distributions become more regular, i.e., coefficient of variation decreases, the performance measures of the queueing system usually improve.