# Successive Approximation Method

## Quadratic Criterion & Linear Dynamics

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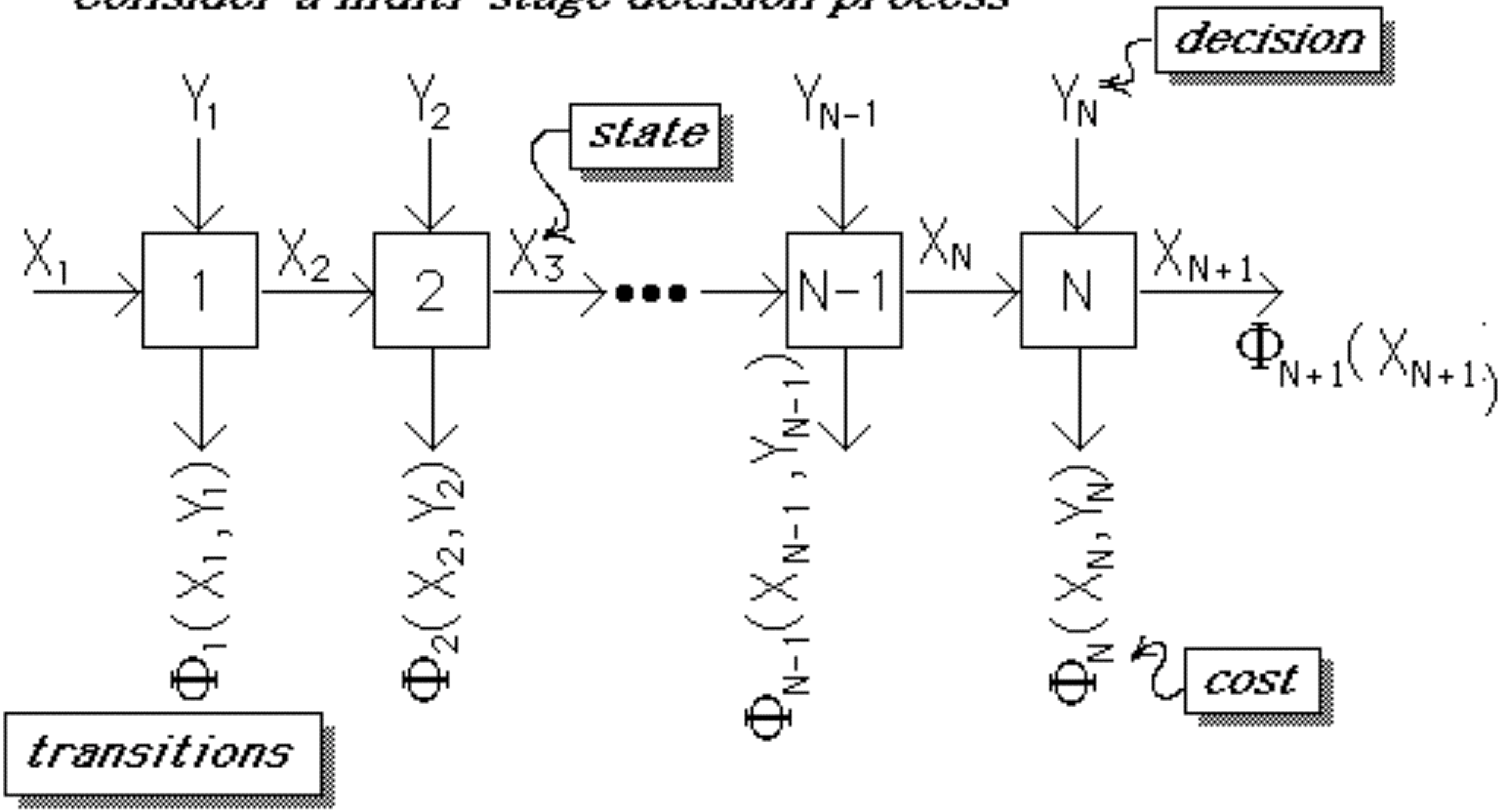
# *Consider a multi-stage decision process*




$$X_{i+1} = \Psi_i(X_i, Y_i)$$

$$\textit{Minimize} \; \sum_{i=1}^{N} \Phi_i\left(X_i, Y_i\right)$$

$$\text{subject to } \; X_{i+1} = \Psi_i\left(X_i, Y_i\right), \;\; i=1, 2, \ldots N$$

If the objective function $\Phi_i$ is quadratic in X and Y, and the transition function $\Psi_i$ is linear in X and Y *(the "QC/LD" case)*, we have a closed-form solution for the problem.

Otherwise, we can try *successively approximating* the problem by a QC/LD problem.

*(Successive approximation of a convex constrained programming problem by a linearly-constrained quadratic programming problem has been a very successful approach in nonlinear programming.)*

## *Successive Approximation Algorithm (SAM)*

Step 0: "Guess" at a decision sequence $\bar{Y}_1, \bar{Y}_2, \ldots \bar{Y}_N$

Step 1: Use the transition functions together with the initial state $X_1$ to compute a trajectory $\bar{X}_1, \bar{X}_2, \bar{X}_3, \ldots \bar{X}_N$

Step 2: Compute $\Phi_i(\bar{X}_i, \bar{Y}_i)$ , its gradient $\nabla\Phi_i(\bar{X}_i, \bar{Y}_i)$

i.e., $\frac{\partial}{\partial X_i}\Phi_i(\bar{X}_i, \bar{Y}_i), \frac{\partial}{\partial Y_i}\Phi_i(\bar{X}_i, \bar{Y}_i)$

and its Hessian matrix $\nabla^2\Phi_i(\bar{X}_i, \bar{Y}_i)$ , i.e.,

$$\frac{\partial^2}{\partial X_i^2}\Phi_i(\bar{X}_i, \bar{Y}_i), \frac{\partial^2}{\partial X_i \partial Y_i}\Phi_i(\bar{X}_i, \bar{Y}_i), \ \& \frac{\partial^2}{\partial Y_i^2}\Phi_i(\bar{X}_i, \bar{Y}_i)$$

*(SAM, continued)*

Step 3: Approximate the cost at each stage by the ***Taylor series*** expansion about up to & including the quadratic terms:

$$\Phi_i(X_i,Y_i) \approx \Phi_i(\overline{X}_i,\overline{Y}_i) + \frac{\partial\Phi}{\partial X_i}(\overline{X}_i,\overline{Y}_i)(X_i - \overline{X}_i) + \frac{\partial\Phi}{\partial Y_i}(\overline{X}_i,\overline{Y}_i)(Y_i - \overline{Y}_i)$$

$$+ \frac{1}{2}\left[\frac{\partial^2\Phi}{\partial X_i^2}(\overline{X}_i,\overline{Y}_i)(X_i - \overline{X}_i)^2 + \frac{\partial^2}{\partial X_i \partial Y_i}\Phi_i(\overline{X}_i,\overline{Y}_i)(X_i - \overline{X}_i)(Y_i - \overline{Y}_i)\right.$$

$$\left. + \frac{\partial^2\Phi}{\partial Y_i^2}(\overline{X}_i,\overline{Y}_i)(Y_i - \overline{Y}_i)^2\right]$$

*(SAM, continued)*

Step 4: Approximate the transition function by a linear function:

$$X_{i+1} = \Psi_i (X_i, Y_i) \approx \Psi_i (\overline{X}_i, \overline{Y}_i) + \frac{\partial \Psi}{\partial X_i}\left(\overline{X}_i, \overline{Y}_i\right)\left(X_i - \overline{X}_i\right)$$

$$+ \frac{\partial \Psi}{\partial Y_i}\left(\overline{X}_i, \overline{Y}_i\right)\left(Y_i - \overline{Y}_i\right)$$

*(SAM, continued)*

Step 5: Use the closed-form solution to the QC/LD problem to compute the optimal decisions

$\hat{Y}_1, \hat{Y}_2, \hat{Y}_3, \ldots \hat{Y}_N$

Step 6: Use the transition functions, together with initial state $X_1$ and decisions $\hat{Y}_1, \hat{Y}_2, \hat{Y}_3, \ldots \hat{Y}_N$ to compute the new trajectory

$$\hat{\mathbf{X}}_{i+1} = \boldsymbol{\Psi}_i\left(\hat{\mathbf{X}}_i, \hat{\mathbf{Y}}_i\right) \quad , i=1,2,3,\ldots N$$

Step 7: If the termination criterion is not satisfied, let $\bar{X} = \hat{X}$ and $\bar{Y} = \hat{Y}$, and return to step 2.

**EXAMPLE**

$$\text{Minimize} \quad Y_1^4 + X_2^4 + Y_2^4 + X_3^4$$

subject to

$$\begin{cases} X_{i+1} = X_1^2 + 4\,Y_i \ , \ i=1,2 \\ X_1 = 2 \end{cases}$$

$$\overline{Y}_1 = -1 \ , \ \overline{Y}_2 = -0.1$$

# *APL Code for the Criterion Function*

Number of stages: N = 2

Objective Function

```
Z←T FN XY
⍝
⍝         Objective Function for SAMDP
⍝
Z←(0 1 1)[T]×XY[1]*4
→0 IF N<T
Z←Z+(1 1 0)[T]×XY[2]*4
```

## *APL Code for Gradient of Criterion Function*

```
G←T GRADF XY
⍝
⍝     Gradient of objective function for example problem
⍝
G←(0 1 1)[T]×4×XY[1]*3
→Last IF N<T   ⋄ G←G,0  ⋄ →0
Last:G←G,(1 1 0)[T]×4×XY[2]*3
```

# *APL Code for Hessian Matrix of Criterion Function*

```
H←T HESSIAN XY
⍝
⍝      Hessian matrix for objective function
⍝                   of example problem
⍝
H←2 2⍴0
H[1;1]←(0 1 1)[T]×12×XY[1]*2
→0 IF N<T
H[2;2]←(1 1 0)[T]×12×XY[2]*2
```

## *APL Code for Transition Function*

```
Z←T PHI XY
⍝
⍝       Transition function for example problem
⍝
Z←(XY[1]*2) + 4×XY[2]
```

## *APL Code for Gradient of Transition  Function*

```
G←T GRADPHI XY
⍝
⍝     Gradient of transition function for example problem
⍝
G←(2×XY[1]),4
```

# ITERATION #1

QC/LD Approximation at:

| i | 1 | 2 | 3 |
|---|---|---|---|
| X[i] | 2 | 0 | ¯0.4 |
| Y[i] | ¯1 | ¯0.1 | 0 |

Objective function value is 1.0257

| T | A | B | C | D | E | F | G | H | K |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 6 | 0 | 12 | 7 | 4 | 4 | ¯4 |
| 1 | 0 | 0 | 0.06 | 0 | 0.012 | 0.0007 | 0 | 4 | 0 |
| 2 | 0.96 | 0 | 0 | 0.512 | 0 | 0.0768 | 0 | 0 | 0 |

Solution of QC/LD Approximation is:

| i | 1 | 2 | 3 |
|---|---|---|---|
| X[i] | 2 | 0 | ¯0.267185 |
| Y[i] | ¯1 | ¯0.0667964 | 0 |

Sum of |Y-Y'| = 0.0332036

# Iteration 2

QC/LD Approximation at:

| i | 1 | 2 | 3 |
|---|---|---|---|
| X[i] | 2 | 0 | ¯0.267185 |
| Y[i] | ¯1 | ¯0.0667964 | 0 |

Objective function value is 1.00512

| A | B | C | D | E | F | G | H | K |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 6 | 0 | 12 | 7 | 4 | 4 | ¯4 |
| 0 | 0 | 0.02677 | 0 | 0.003576 | 0.0001393 | 0 | 4 | 0 |
| 0.428328 | 0 | 0 | 0.152591 | 0 | 0.0152888 | 0 | 0 | 0 |

Solution of QC/LD Approximation is:

| i | 1 | 2 | 3 |
|---|---|---|---|
| X[i] | 2 | 0 | ¯0.17847 |
| Y[i] | ¯1 | ¯0.0446175 | 0 |

Sum of |Y-Y'| = 0.0221788

QC/LD Approximation at:

Iteration 3

| | | | |
|---|---|---|---|
| X[i] | 2 | 0 | ¯0.17847 |
| Y[i] | ¯1 | ¯0.0446175 | 0 |

Objective function value is 1.00102

| T | A | B | C | D | E | F | G | H | K |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0E0 | 0 | 6E0 | 0E0 | 1.2E1 | 7E0 | 4 | 4 | ¯4 |
| 1 | 0E0 | 0 | 1.19444E¯2 | 0E0 | 1.06586E¯3 | 2.77409E¯5 | 0 | 4 | 0 |
| 2 | 1.9111E¯1 | 0 | 0E0 | 4.54765E¯2 | 0E0 | 3.04358E¯3 | 0 | 0 | 0 |

Solution of QC/LD Approximation is:

| i | 1 | 2 | 3 |
|---|---|---|---|
| X[i] | 2 | 0 | ¯0.119212 |
| Y[i] | ¯1 | ¯0.0298029 | 0 |

Sum of |Y-Y'| = 0.0148146

QC/LD Approximation at:

| | | | |
|---|---|---|---|
| X[i] | 2 | 0 | ¯0.119212 |
| Y[i] | ¯1 | ¯0.0298029 | 0 |

Objective function value is 1.0002

**Iteration 4**

| T | A | B | C | D | E | F | G | H | K |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0E0 | 0 | 6E0 | 0E0 | 1.2E1 | 7E0 | 4 | 4 | ¯4 |
| 1 | 0E0 | 0 | 5.32928E¯3 | 0E0 | 3.17656E¯4 | 5.52246E¯6 | 0 | 4 | 0 |
| 2 | 8.52684E¯2 | 0 | 0E0 | 1.35533E¯2 | 0E0 | 6.05892E¯4 | 0 | 0 | 0 |

Solution of QC/LD Approximation is:

| i | 1 | 2 | 3 |
|---|---|---|---|
| X[i] | 2 | 0 | ¯0.079629 |
| Y[i] | ¯1 | ¯0.0199073 | 0 |

Sum of |Y-Y'| = 0.00989565

Since $\sum_{i}\left|Y - Y'\right|$ is less than the tolerance (0.01), the algorithm terminates.