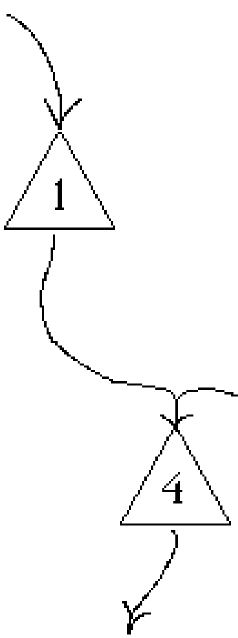
Optimal Control of a Network of Reservoirs



A Discrete-time Dynamic Programming algorithm, with

- Quadratic Criterion and
- Linear Dynamics

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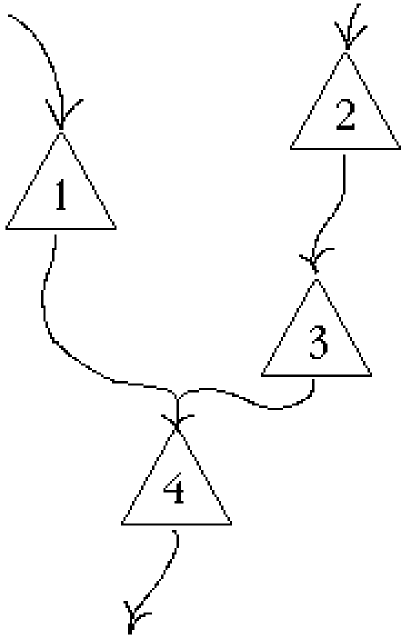
Sample reservoir data

reservoirs = 4
time periods = 8

Reservoir adjacency matrix

from	to			
	1	2	3	4
1	0	0	0	1
2	0	0	1	0
3	0	0	0	1
4	0	0	0	0



List of reservoirs directly upstream of each reservoir #i:

i	upstream
1	
2	
3	2
4	1 3

Storage Targets

stage	reservoir			
	1	2	3	4
1	90	75	100	200
2	100	80	110	190
3	80	70	90	195
4	90	75	100	180
5	90	75	100	200
6	100	80	110	190
7	80	70	90	195
8	90	75	100	180
9	90	75	100	200

Penalty for Storage Deviation
1.00 for all stages & reservoirs

Release Targets

stage	reservoir			
	1	2	3	4
1	100	75	100	250
2	90	70	90	220
3	110	80	110	250
4	100	75	105	225
5	100	75	100	250
6	90	70	90	220
7	110	80	110	250
8	100	75	105	225

Penalty for Release Deviation
1.00 for all stages & reservoirs

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Storage Targets
200
150
100
50
0
2
4
6
8
Stage
reservoir 1
reservoir 2
reservoir 3
reservoir 4

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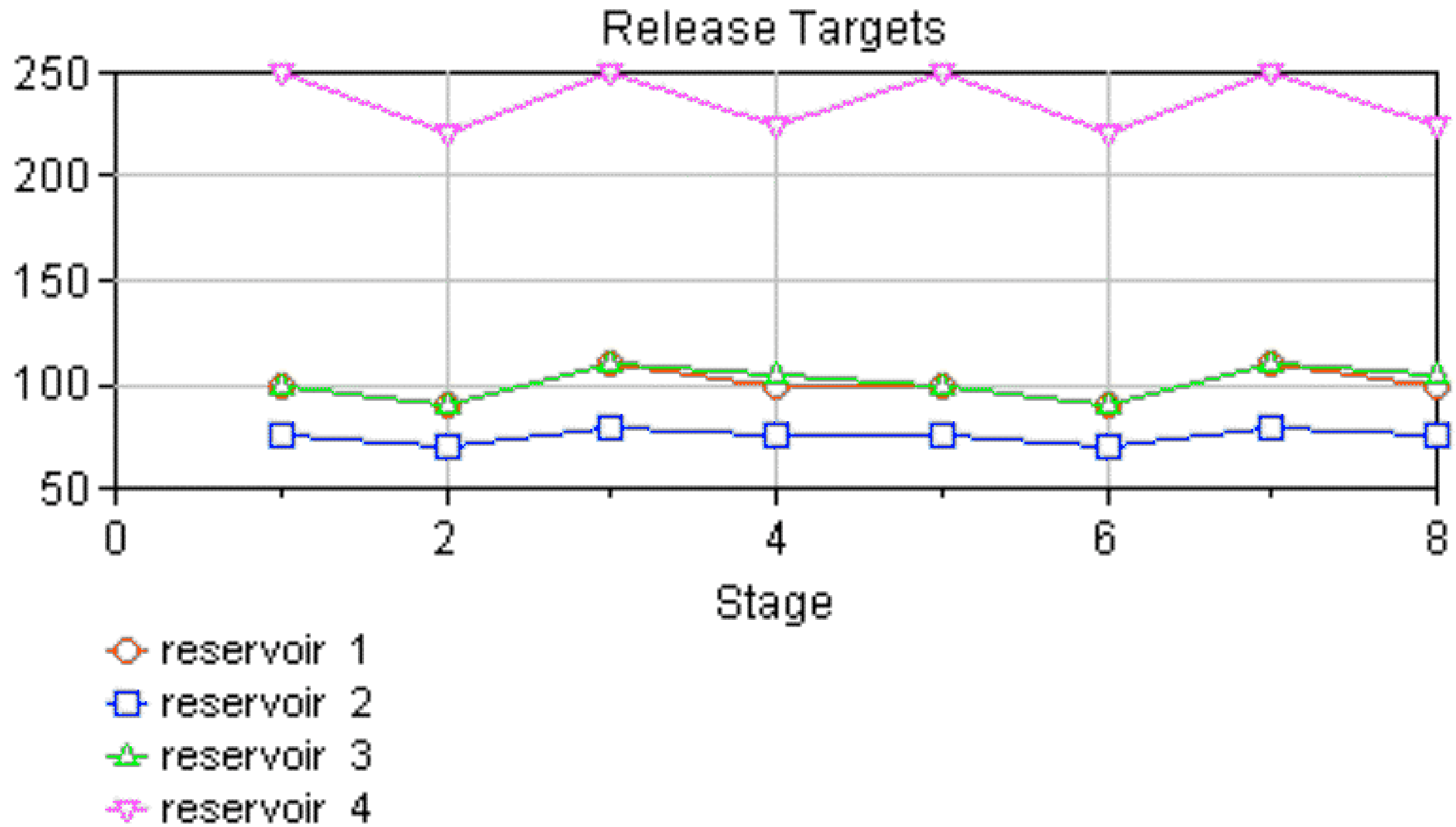
Release Targets
250
200
150
100
50
0
2
4
6
8
Stage
reservoir 1
reservoir 2
reservoir 3
reservoir 4

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Expected Inflows

stage	reservoir			
	1	2	3	4
1	100	75	100	240
2	90	70	90	225
3	100	75	100	240
4	105	85	105	250
5	100	75	100	240
6	90	70	90	225
7	100	75	100	240
8	105	85	105	250

Std Deviation of Inflows

stage	reservoir			
	1	2	3	4
1	7	8	8	10
2	9	3	4	4
3	6	6	9	5
4	7	8	8	10
5	9	3	4	4
6	6	6	9	5
7	7	8	8	10
8	9	3	4	4

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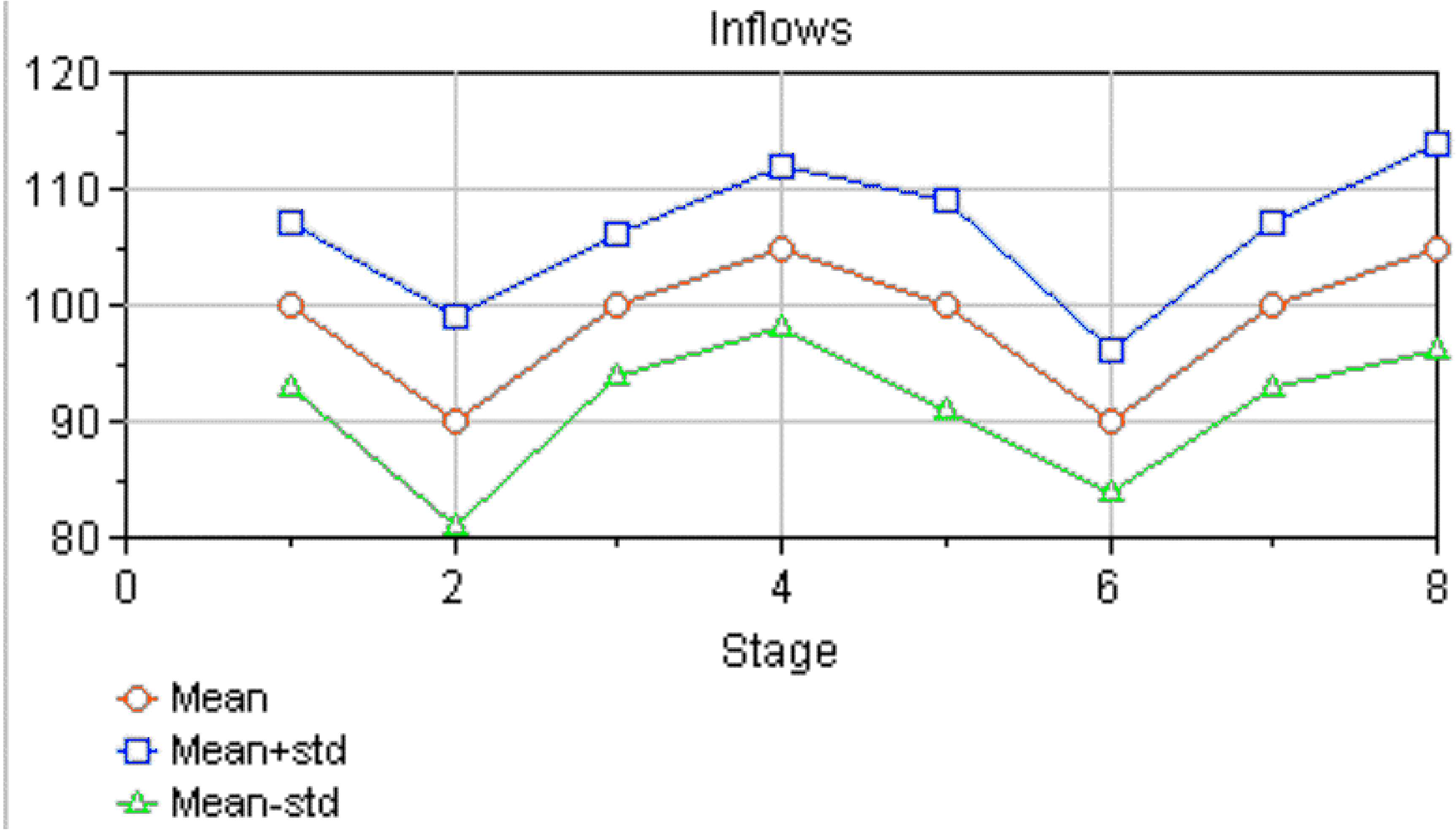
Example: Reservoir #1 Inflows



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Evaporation Rates

stage	reservoir			
	1	2	3	4
1	0.05	0.02	0.03	0.08
2	0.05	0.02	0.03	0.08
3	0.05	0.02	0.03	0.08
4	0.05	0.02	0.03	0.08
5	0.05	0.02	0.03	0.08
6	0.05	0.02	0.03	0.08
7	0.05	0.02	0.03	0.08
8	0.05	0.02	0.03	0.08

Capacities

reservoir			
1	2	3	4
125	150	125	250

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Reservoir Release Planning Problem: Sample reservoir data

of stages (including final stage) = 8
# of state variables/stage = 4
# of decision variables/stage = 4

A = cost of X*2

all stages

	state			
state	1	2	3	4
1	1	0	0	0
2	0	1	0	0
3	0	0	1	0
4	0	0	0	1

B = cost of XY

all stages

	decision			
state	1	2	3	4
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0
4	0	0	0	0

Quadratic Criterion

C = cost of Y*2

all stages

decid	decision			
	1	2	3	4
1	1	0	0	0
2	0	1	0	0
3	0	0	1	0
4	0	0	0	1

Quadratic Criterion

D = cost of X

stage	state			
	1	2	3	4
1	180	150	200	400
2	200	160	220	380
3	160	140	180	390
4	180	150	200	360
5	180	150	200	400
6	200	160	220	380
7	160	140	180	390
8	180	150	200	360
9	180	150	200	400

E = cost of Y

stage	decision			
	1	2	3	4
1	¯200	¯150	¯200	¯500
2	¯180	¯140	¯180	¯440
3	¯220	¯160	¯220	¯500
4	¯200	¯150	¯210	¯450
5	¯200	¯150	¯200	¯500
6	¯180	¯140	¯180	¯440
7	¯220	¯160	¯220	¯500
8	¯200	¯150	¯210	¯450

F = constant in cost

stage	value
1	151850
2	134100
3	150525
4	133400
5	151850
6	134100
7	150525
8	133400
9	63725

Quadratic Criterion

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G = coefficient of X

all stages

	state			
state	1	2	3	4
1	0.95	0	0	0
2	0	0.98	0	0
3	0	0	0.97	0
4	0	0	0	0.92

H = coefficient of Y

all stages

	decision			
state	1	2	3	4
1	¯1	0	0	0
2	0	¯1	0	0
3	0	1	¯1	0
4	1	0	1	¯1

Linear Dynamics

K = constants in transition

	stage							
stge	1	2	3	4	5	6	7	8
1	100	90	100	105	100	90	100	105
2	75	70	75	85	75	70	75	85
3	100	90	100	105	100	90	100	105
4	240	225	240	250	240	225	240	250

Backward Computation of Arrays P, q, r, S, U, & t

Let $P_{N+1} = A_{N+1}, q_{N+1} = d_{N+1}$, and $r_{N+1} = f_{N+1}$, and

for $n = N, N-1, \ldots$ 2, 1: compute

$$\begin{cases} S_n = C_n + H_n^t P_{n+1} H_n \\ t_n = e_n + 2H_n^t P_{n+1} k_n + H_n^t q_{n+1} \\ U_n = B_n^t + 2H_n^t P_{n+1} G_n \end{cases}$$

and

$$\begin{cases} P_n = A_n + G_n^t P_{n+1} G_n - \frac{1}{4} U_n^t S_n^{-1} U_n \\ q_n = d_n + 2k_n^t P_{n+1} G_n + q_{n+1} G_n - \frac{1}{2} t_n^t S_n^{-1} U_n \\ r_n = f_n + k_n^t P_{n+1} k_n + q_{n+1} k_n + r_{n+1} - \frac{1}{4} t_n^t S_n^{-1} t_n \end{cases}$$

Forward Computations of Optimal States & Decisions

After the arrays S_n^{-1}, U_n, and t_n have been computed during the ***backward*** pass, a ***forward*** pass is performed in which the states and decisions are recursively computed:

$$\text{for } n = 1, 2, \ldots N \quad \begin{cases} Y_n = -0.5\, S_n^{-1} \left(U_n X_n - t_n\right) \\ X_{n+1} = G_n X_n + H_n Y_n + k_n \end{cases}$$

At this time, of course, the values of the **random variables** (k_n) are used instead of their expected values, after their actual values have been realized.

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Optimal Cost: $437268

Linear Decision Rule: Note that the optimal decision at each stage is given by a linear function of the state at that stage:

Stage 1: $Y^{(1)} = \left(Y_1^{(1)}, Y_2^{(1)}, Y_3^{(1)}, Y_4^{(1)}\right) =$

X_1	X_2	X_3	X_4	Constant
0.51536	-0.033965	-0.0534673	-0.11246	71.4894
-0.0299732	0.508528	-0.165838	-0.0478524	61.0455
-0.0800116	0.338119	0.450053	-0.139234	123.254
0.313601	0.244011	0.291974	0.416158	308.484

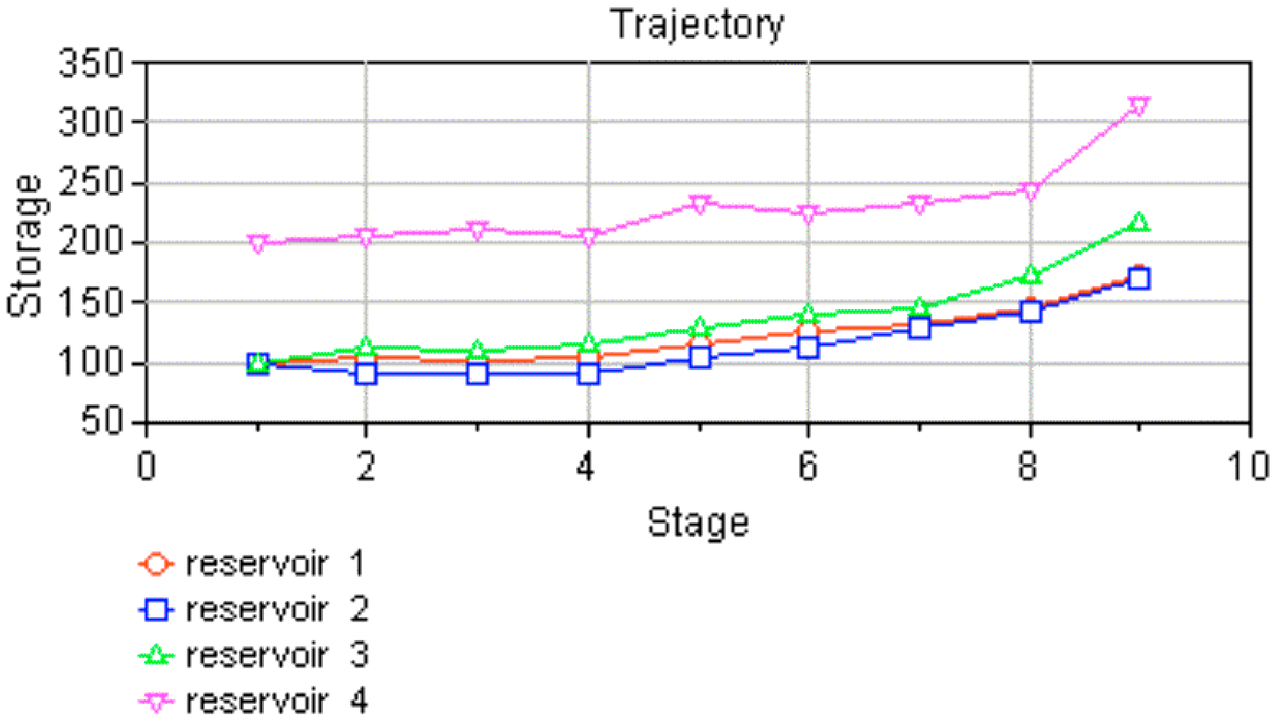
For example, the release from reservoir #1 in period #1 should be

$$Y_1^{(1)} = 0.5154X_1 - 0.034X_2 - 0.0535X_3 - 0.1125X_4 + 71.4894$$

Following are the results (ignoring capacity & nonnegativity constraints).

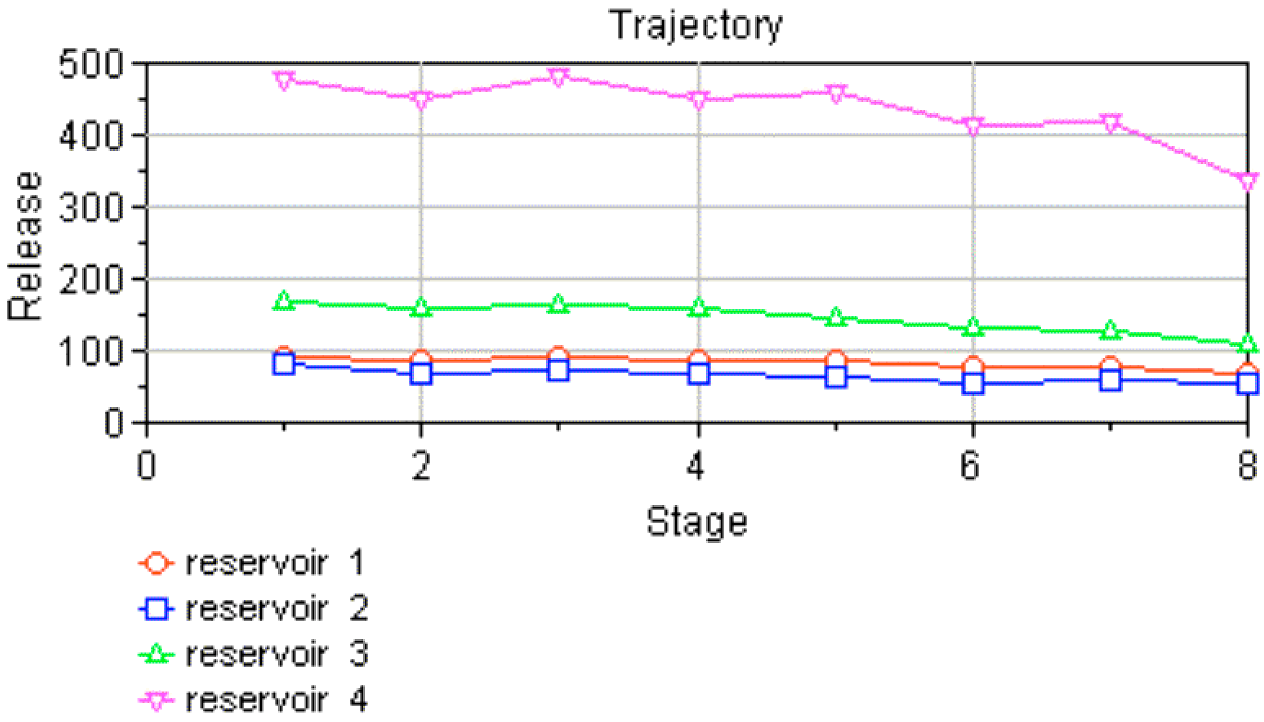
State Variables (Storage Volumes)

t	1	2	3	4
1	100	100	100	200
2	103.21	90.2533	113.524	205.339
3	101.178	90.4544	109.501	209.683
4	103.629	89.5983	116.545	205.39
5	116.65	104.744	128.285	232.13
6	125.616	113.858	140.692	225.077
7	130.999	128.648	145.746	232.607
8	146.104	143.339	171.42	242.801
9	173.654	171.241	217.01	313.51



Decision Variables (Releases)

t	1	2	3	4
1	91.7901	82.7467	166.223	476.674
2	86.8717	67.9939	158.611	449.711
3	92.4899	74.0471	163.718	483.726
4	86.7977	68.0622	157.826	451.452
5	85.2015	63.7913	147.536	461.22
6	78.3363	52.9328	133.658	411.459
7	78.3449	57.7358	127.689	417.231
8	70.1442	54.2314	108.499	338.51



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As a heuristic algorithm, we can impose the capacity constraints during the forward pass:

$$\text{for } n = 1, 2, \ldots N \quad \begin{cases} Y_n = \mathrm{Z_n} + \max\{0, \ -0.5\, S_n^{-1} \left(U_n X_n - t_n\right) \} \\ X_{n+1} = \max\{0, G_n X_n + H_n Y_n + k_n\} \\ Z_{n+1} = \max\left\{0, \ \hat{X} - X_{n+1}\right\} \\ X_{n+1} = \min\left\{\hat{X}, X_{n+1}\right\} \end{cases}$$

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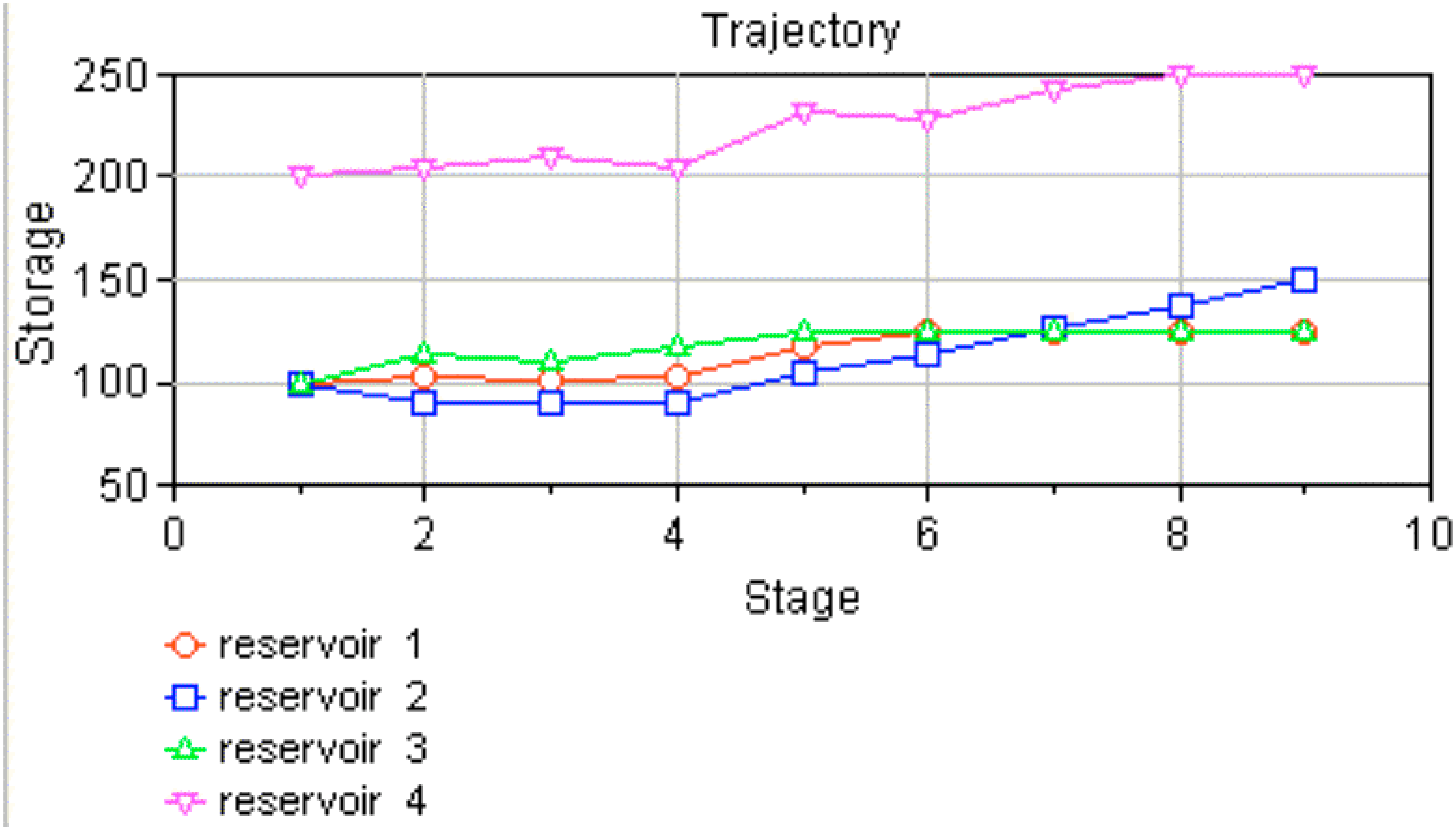
That is,

- optimal releases Y_n are first computed, including any **spill** Z_n.
- any **negative** releases are truncated at zero before computing the storage volumes at the next stage;
- any storage volumes which are **negative** are truncated.
- **spill** Z_n is determined, and storage volumes exceeding capacities $\hat{X}$ are truncated.

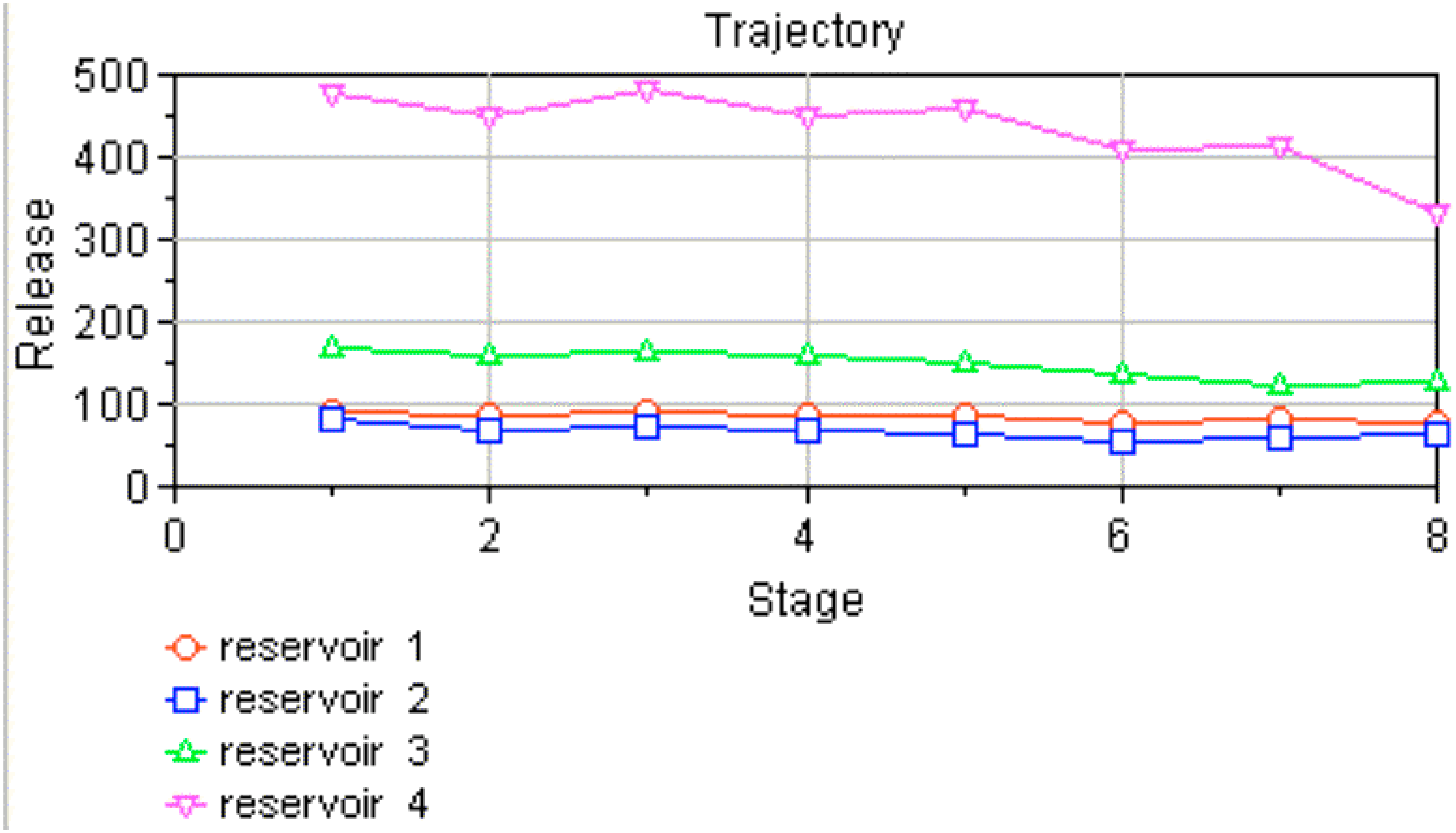
While not guaranteed to be optimal, the solution thus obtained will be **feasible**.

Solution with capacity constraints imposed:

$$\hat{X} = [125,\ 150,\ 125,\ 250]$$



Releases which result in feasible storage volumes:



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Simulation

Number of runs: 25

- **Random inflows** are generated during the forward passes (assuming normal distributions with means & standard deviations as specified).
- **Capacity** constraints imposed

Summary Statistics

Average cost: $389,623

Std Deviation of cost: $12,201

(Note: cost does not include cost of any spill at terminal stage! This results in an "optimal" cost which appears to be less than that of the unconstrained case, e.g., $437,268! To account for this, the cost factor for storage target deviation at stage NN+1=9 should have been made larger.)

State Variables (Storage volumes in reservoirs)

Average values

t	1	2	3	4
1	100	100	100	200
2	104.09	90.3913	114.887	205.149
3	100.559	91.3719	108.941	210.787
4	103.653	89.5824	113.029	204.355
5	115.007	101.128	121.769	231.438
6	121.336	110.759	124.522	229.124
7	121.422	123.708	123.067	242.617
8	124.949	134.691	125	244.468
9	124.594	149.588	125	250

Standard deviations

t	1	2	3	4
1	0	0	0	0
2	7.57496	8.011	7.0349	9.68853
3	9.11988	4.5566	4.63484	4.88583
4	7.33	6.59179	8.97661	5.57855
5	5.88178	6.99187	4.68586	10.4534
6	5.17381	4.81111	1.46544	9.10376
7	5.05679	6.48711	3.80095	7.50218
8	0.251048	6.5838	0	8.23749
9	1.98777	1.38213	0	0

Minimum values

t	1	2	3	4
1	100	100	100	200
2	85.6996	76.0953	102.666	184.278
3	81.8726	81.4095	102.184	199.828
4	89.1696	77.991	94.2232	190.268
5	99.832	88.0986	110.905	206.316
6	107.189	104.419	118.209	212.497
7	105.782	110.374	108.248	228.792
8	123.719	121.041	125	222.3
9	114.856	143.928	125	250

Maximum values

t	1	2	3	4
1	100	100	100	200
2	120.166	105.784	125	223.552
3	115.053	102.858	124.752	222.461
4	116.73	101.543	125	218.614
5	125	120.733	125	250
6	125	125.785	125	245.231
7	125	136.397	125	250
8	125	150	125	250
9	125	150	125	250

Decision Variables (Releases)

Average values

t	1	2	3	4
1	91.7901	82.7467	166.223	476.674
2	87.2685	67.8202	159.604	450.338
3	92.0455	74.5715	163.671	484.05
4	87.1208	68.6962	156.795	450.026
5	85.0843	63.1916	148.625	458.212
6	81.5804	54.2656	134.878	407.091
7	77.4393	59.718	123.24	414.664
8	80.4255	61.8473	124.405	334.228

Standard deviations

t	1	2	3	4
1	0	0	0	0
2	3.90599	4.42924	5.07352	5.08147
3	4.58772	2.13054	3.13399	4.60048
4	3.91195	3.50932	5.7221	4.43153
5	3.67528	3.51012	7.61355	6.01262
6	7.72275	2.24016	8.05432	3.92196
7	6.7497	2.92701	8.9507	4.96378
8	7.62037	2.92289	12.777	7.62017

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Minimum values

t	1	2	3	4
1	91.7901	82.7467	166.223	476.674
2	80.3985	58.3352	150.926	434.948
3	81.1912	69.6975	159.38	472.917
4	79.1201	61.6079	148.526	442.18
5	76.121	55.2098	137.635	447.282
6	70.2548	51.6564	119.296	398.99
7	65.8775	53.5703	107.814	407.473
8	64.0001	56.2728	93.275	322.604

Maximum values

t	1	2	3	4
1	91.7901	82.7467	166.223	476.674
2	96.979	75.513	169.837	459.171
3	100.046	79.9238	172.89	491.816
4	92.9452	74.2467	173.984	457.826
5	92.9278	72.6012	162.519	473.375
6	99.7614	61.4332	154.114	414.304
7	92.5297	66.236	143.317	428.425
8	96.5024	71.6386	148.669	353.614

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Count of Storage Volumes Reaching Capacity (of 25 runs)

t	1	2	3	4
1	0	0	0	0
2	0	0	4	0
3	0	0	0	0
4	0	0	2	0
5	2	0	15	3
6	15	0	22	0
7	10	0	15	7
8	24	1	25	14
9	24	22	25	25

Count of Storage Volumes Reaching Zero

<none>

Count of Releases Reaching Zero

<none>

```
Number of runs: 25
Inflows randomly generated during forward pass
Constraints imposed: none
```

Summary Statistics

Average cost: $442,413
Std Deviation of cost: $19,413

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State Variables (Storage Volumes)

Average values

t	1	2	3	4
1	100	100	100	200
2	103.765	92.7689	112.414	206.432
3	103.408	91.764	110.869	209.101
4	105.261	89.3145	116.393	205.046
5	121.588	106.418	130.489	233.006
6	125.936	114.432	141.229	224.159
7	132.369	128.597	143.288	232.309
8	147.025	142.865	171.25	242.113
9	172.794	171.593	216.956	313.535

Standard deviations

t	1	2	3	4
1	0	0	0	0
2	6.52688	8.18039	7.47952	10.2696
3	8.24309	6.02375	3.31642	6.3006
4	7.45871	7.73781	9.04458	5.35308
5	7.41475	10.3543	8.80079	7.48693
6	8.37643	6.24189	5.18972	4.273
7	5.82488	4.71432	11.8728	5.56372
8	6.09231	8.67604	10.8749	9.25183
9	12.0787	7.5115	7.14424	6.20118

Minimum values

t	1	2	3	4
1	100	100	100	200
2	93.3472	71.333	94.3099	182.958
3	89.8635	75.6479	103.116	196.487
4	83.4945	73.4938	98.7217	192.787
5	103.584	90.601	107.643	217.063
6	109.238	104.381	133.068	216.014
7	119.224	118.48	117.867	221.703
8	135.299	126.236	152.251	222.584
9	150.794	158.869	203.148	299.28

Maximum values

t	1	2	3	4
1	100	100	100	200
2	119.051	104.398	122.918	235.116
3	123.731	101.692	118.068	223.391
4	118.287	102.68	134.028	214.504
5	133.216	133.229	142.428	245.924
6	146.924	133.071	152.987	232.246
7	143.721	137.739	161.259	245.349
8	157.973	159.28	190.697	258.482
9	202.736	189.584	233.043	327.88

Decision Variables (Releases)

Average values

t	1	2	3	4
1	91.7901	82.7467	166.223	476.674
2	87.0084	69.3873	158.764	450.628
3	93.5851	74.4432	164.672	484.892
4	87.6943	67.9097	157.577	451.704
5	87.4443	64.0389	148.49	464.051
6	78.5506	53.1557	134.172	411.44
7	79.2118	58.1598	126.62	416.934
8	70.6387	54.1013	108.409	338.396

Standard deviations

t	1	2	3	4
1	0	0	0	0
2	3.34815	3.85125	5.5764	5.87468
3	4.07318	2.86621	3.15585	4.90513
4	3.56408	4.39316	4.22289	5.64459
5	4.12816	4.6959	6.49299	5.00762
6	4.30749	2.56764	3.93087	3.92811
7	2.80889	2.58317	5.35298	4.31436
8	2.83072	3.55129	3.94945	3.59567

Minimum values

t	1	2	3	4
2	81.8286	61.2653	142.751	438.357
4	77.3382	58.3574	149.067	441.159
6	71.8092	49.0762	127.494	400.067
8	64.3853	47.5246	101.539	332.237

Maximum values

t	1	2	3	4
2	93.64	75.2392	168.65	467.938
4	92.6913	74.5417	165.03	464.362
6	88.8727	60.6325	141.932	419.98
8	75.6702	62.226	114.768	345.354

Count of Storage Volumes Reaching Capacity

```
t  1  2  3  4
1  0  0  0  0
2  0  0  0  0
3  0  0  0  0
4  0  0  5  0
5  7  0 20  0
6 13  0 25  0
7 23  0 23  0
8 25  6 25  5
9 25 25 25 25
```

Count of Storage Volumes Reaching Zero

<none>

Count of Releases Reaching Zero

<none>