Poisson Processes

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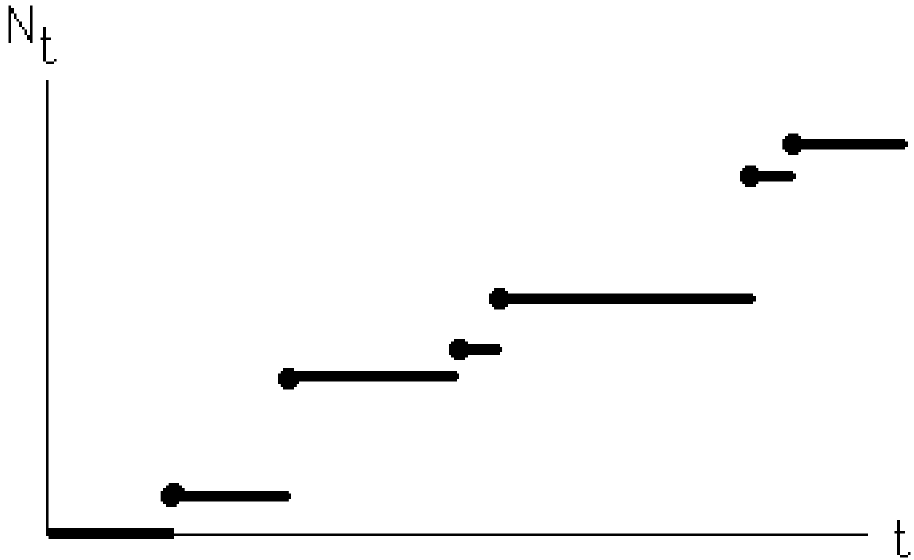
Arrival Process

a stochastic process $\{N_t;\ t \geq 0\}$

such that

- N_t is non-decreasing
- N_t increases by jumps only
- N_t is right-continuous
- $N_0 = 0$


N_t
t

Poisson Process

An arrival process $\{N_t; t \geq 0\}$ such that:

- each jump is of magnitude **1**
- for any $s,t \geq 0$, the difference $N_{s+t} - N_t$ is independent of past history $\{N_u; u \leq t\}$
- for any $s,t \geq 0$, the distribution of $N_{s+t} - N_t$ is independent of t

For any $s,t \geq 0$, the distribution of $N_{s+t} - N_t$ is

$$P\{ N_{s+t} - N_t = k\} = e^{-\lambda s} (\lambda s)^k / k!$$

i.e. the distribution of number of arrivals in *any* interval of length **s** is **Poisson**.

The parameter λ is the **arrival rate.**

Time of Arrivals

Let $\{N_t;\ t \geq 0\}$ be a Poisson process.

Define another (discrete-parameter, continuous-value) stochastic process

$$\{T_k;\ k = 1, 2, \ldots\}$$

by T_k = time of jump #k of N_t.

For any $n \geq 0$, the distribution of the time between arrival #n and #n+1 is

$$P\{T_{n+1} - T_n \leq t \mid T_1, T_2, \ldots T_n\} =$$

$$= P\{ T_1 \leq t \} = 1 - e^{-\lambda t}, \; t \geq 0$$

(exponential distribution)

Memorylessness of the exponential distribution

$$P\{T_1 > t+s \mid T_1 > t\} = P\{T_1 > s\}, \quad s,t \geq 0$$

that is, knowing that an interarrival time has already lasted **t** time units does not alter the probability of its lasting another **s** time units.

Superposition of Poisson Processes

Let $\{L_t; t \geq 0\}$ and $\{M_t; t \geq 0\}$ be two poisson processes with arrival rates λ and μ, respectively.

The **superposition** of processes L_t and M_t is a new stochastic process N_t defined by $N_t = L_t + M_t$

N_t is itself a **Poisson process** with arrival rate $\nu = \lambda + \mu$.

Decomposition of a Poisson process

Let $\{N_t;\ t \geq 0\}$ be a **Poisson** process with rate λ.

Let $\{X_n;\ n = 1,2,\ldots\}$ be a **Bernoulli** process with probability **p** of success at each trial, and let S_n = # successes in first n trials.

Suppose that the n^{th} trial is performed at the time T_n of the n^{th} arrival, and define two new stochastic processes M_t (#successes) and L_t (# failures) by

$$M_t = S_{N_t},$$

and
$$L_t = N_t - M_t$$

The processes M_t and L_t are **Poisson** with arrival rates **$p\lambda$** and $(1-p)\lambda$, respectively.

Compound Poisson Process

In a Poisson process, the jumps must be of magnitude **1**; in the Compound Poisson process, jumps may be of any size (positive &/or negative). The magnitudes of successive jumps must be i.i.d. random variables, independent of **t**.

Example:

Arrivals of customers into a store form a Poisson process, and the amount of money spent by the n^{th} customer is a random variable X_n. The stochastic process $\{Y_t;\ t \geq 0\}$ defined by

$$Y_t = \text{total sales during } (0,t]$$

is a compound Poisson process.