# Example: Primal-Dual Interior Point Algorithm

This Hypercard stack was prepared by:
Dennis L. Bricker,
Dept. of Industrial Engineering,
University of Iowa,
Iowa City, Iowa 52242
e-mail: dbricker@icaen.uiowa.edu

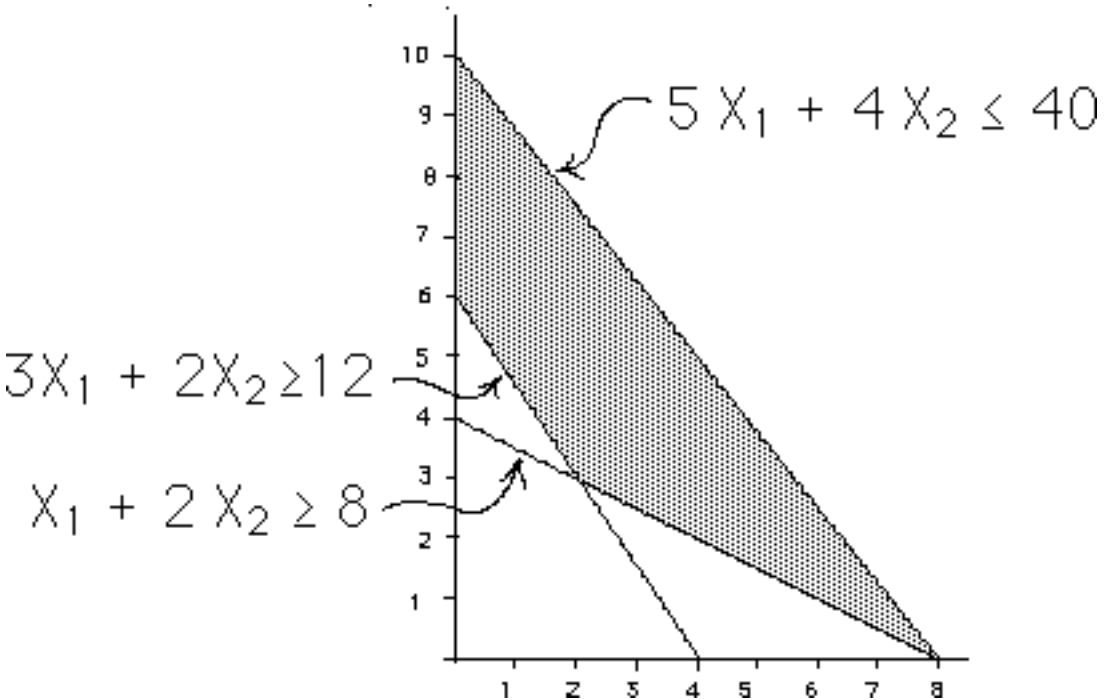
# Example

Minimize $6X_1 + 5X_2$

subject to

$$\begin{cases} 3X_1 + 2X_2 \geq 12 \\ X_1 + 2X_2 \geq 8 \\ 5X_1 + 4X_2 \leq 40 \end{cases}$$

$$X_1 \geq 0, X_2 \geq 0$$


$5X_1 + 4X_2 \leq 40$
$3X_1 + 2X_2 \geq 12$
$X_1 + 2X_2 \geq 8$
10
9
8
7
6
5
4
3
2
1
1
2
3
4
5
6
7
8

*Convert constraints to equalities:*

$$\text{Minimize } 6X_1 + 5X_2$$

subject to

$$\begin{cases} 3X_1 + 2X_2 - X_3 = 12 \\ X_1 + 2X_2 - X_4 = 8 \\ 5X_1 + 4X_2 + X_5 = 40 \end{cases}$$

$$X_1 \geq 0, X_2 \geq 0, X_3 \geq 0, X_4 \geq 0, X_5 \geq 0$$

## *Dual Problem*

Maximize $12\,Y_1 + 8\,Y_2 + 40\,Y_3$

subject to

$$\begin{cases} 3\,Y_1 + Y_2 + 5\,Y_3 \le 6 \\ 2\,Y_1 + 2\,Y_2 + 4\,Y_3 \le 5 \\ -\,Y_1 \le 0 \\ -\,Y_2 \le 0 \\ Y_3 \le 0 \end{cases}$$

$Y_1$, $Y_2$, & $Y_3$ unrestricted in sign

Maximize $12\,Y_1 + 8\,Y_2 + 40\,Y_3$

subject to

*Dual in equality form*

$$\begin{cases} 3\,Y_1 + Y_2 + 5\,Y_3 + Z_1 = 6 \\ 2\,Y_1 + 2\,Y_2 + 4\,Y_3 + Z_2 = 5 \\ -\,Y_1 + Z_3 = 0 \\ -\,Y_2 + Z_4 = 0 \\ Y_3 + Z_5 = 0 \end{cases}$$

$Y_1$, $Y_2$, & $Y_3$ unrestricted in sign

$Z_1 \geq 0, Z_2 \geq 0, Z_3 \geq 0, Z_4 \geq 0, Z_5 \geq 0$

## Complementary Slackness Theorem:

If X is feasible in the primal, and
Y,Z is feasible in the dual problem,
and if

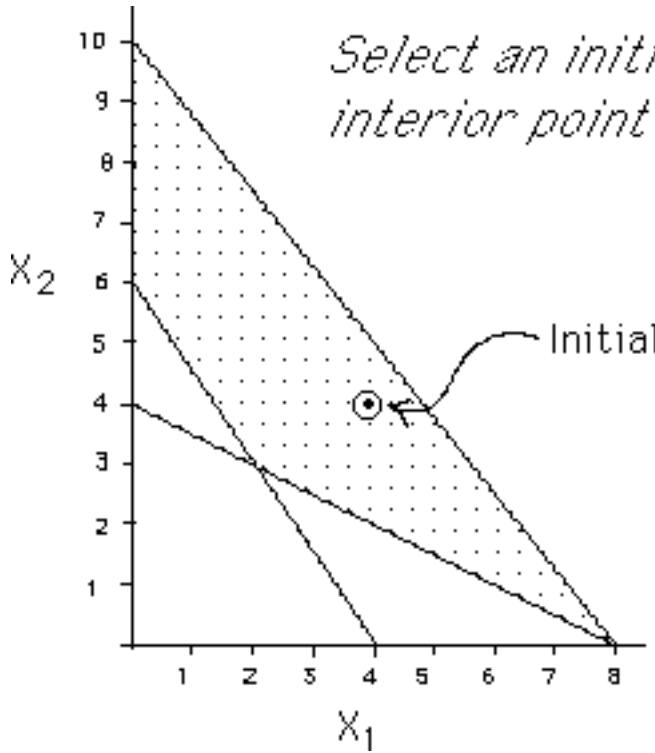
$X_i Z_i = 0$ for i=1,2,3,4, & 5

*i.e., for each i, either $X_i$ or $Z_i$ (or both) must be zero*

Then X, Y, Z are optimal.

## Algorithm Parameters

0.001 = Epsilon (convergence tolerance)
0.01 = alpha (shrinkage factor for mu)
0.995 = tau (stepsize factor)
10 = max number of iterations




*Select an initial primal-feasible interior point (i.e., $X>0$):*

Initial point: $X_1 = 4$
$X_2 = 4$

$\Longrightarrow$ $X_3 = 8$
$X_4 = 4$
$X_5 = 4$

*slack variables*

*Select an initial dual-feasible solution with Z>0*

$$\begin{cases} 3Y_1 + Y_2 + 5Y_3 + Z_1 = 6 \\ 2Y_1 + 2Y_2 + 4Y_3 + Z_2 = 5 \\ -Y_1 + Z_3 = 0 \\ -Y_2 + Z_4 = 0 \\ Y_3 + Z_5 = 0 \end{cases}$$

$Y_1$, $Y_2$, & $Y_3$ unrestricted in sign

$Z_1 \geq 0, Z_2 \geq 0, Z_3 \geq 0, Z_4 \geq 0, Z_5 \geq 0$

Choose initial dual solution: $Y_1 = Y_2 = +1, Y_3 = -1$

$Z_1 = 7, Z_2 = 5, Z_3 = Z_4 = Z_5 = +1$

## Iteration # 1

cx = 44, by = ⁻20, +/xz = 64, gap = 64

Infeasibilities:
primal (dp): 0 0 0
dual (dd): 0 0 0 0 0

| i | x | Δx | x | z | Δz | z | xz |
|---|---|---|---|---|---|---|---|
| 1 | 4 | ⁻1.71974 | 0.0390203 | 7 | ⁻0.790452 | 5.62717 | 0.219574 |
| 2 | 4 | 0.851613 | 5.96147 | 5 | ⁻2.86452 | 0.025 | 0.149037 |
| 3 | 8 | ⁻3.456 | 0.04 | 1 | 1.032 | 2.79234 | 0.111694 |
| 4 | 4 | ⁻0.0165161 | 3.96196 | 1 | 2.20413 | 4.82806 | 19.1286 |
| 5 | 4 | 5.19226 | 15.959 | 1 | 0.901935 | 2.56645 | 40.9581 |

| j | yo | Δy | y |
|---|---|---|---|
| 1 | 1 | 1.032 | 2.79234 |
| 2 | 1 | 2.20413 | 4.82806 |
| 3 | -1 | -0.901935 | -2.56645 |

```
Stepsize: (primal) 2.30324
          (dual) 1.73677

updated mu= 0.60567
gap= 60.567
```

## Iteration # 2

cx = 30.0415, by = -30.5255, +/xz = 60.567, gap = 60.567

Infeasibilities:

primal (dp): 0 4.44089E-15 -7.10543E-15

dual (dd): 0 0 0 0 0

| i | xo | Δx | x |
|---|---|---|---|
| 1 | 0.0390203 | 0.114609 | 7.30237 |
| 2 | 5.96147 | -0.080621 | 0.852094 |
| 3 | 0.04 | 0.182584 | 11.6113 |
| 4 | 3.96196 | -0.0466335 | 1.00655 |
| 5 | 15.959 | -0.250559 | 0.0797951 |

| i | zo | Δz | z | xz |
|---|---|---|---|---|
| 1 | 5.62717 | -6.63311 | 0.0281358 | 0.205458 |
| 2 | 0.025 | 0.0769355 | 0.0899415 | 0.0766386 |
| 3 | 2.79234 | -0.396521 | 2.45764 | 28.5364 |
| 4 | 4.82806 | -4.61836 | 0.929686 | 0.935779 |
| 5 | 2.56645 | -2.48821 | 0.466148 | 0.0371964 |

| j | yo | Δy | y |
|---|---|---|---|
| 1 | 2.79234 | -0.396521 | 2.45764 |
| 2 | 4.82806 | -4.61836 | 0.929686 |
| 3 | -2.56645 | 2.48821 | -0.466148 |

```
Stepsize: (primal) 63.3752
          (dual) 0.844103

updated mu= 0.297914
gap= 29.7914
```

## Iteration # 3

cx = 48.0747, by = 18.2832, +/xz = 29.7914, gap = 29.7914

Infeasibilities:

primal (dp): -3.1493E-11 -2.64926E-11 -8.81215E-11

dual (dd): 8.88178E-16 0 0 0 0

| i | xo | Δx | x |
|---|---|---|---|
| 1 | 7.30237 | -0.492754 | 0.0365118 |
| 2 | 0.852094 | 0.409379 | 6.88855 |
| 3 | 11.6113 | -0.659504 | 1.88663 |
| 4 | 1.00655 | 0.326003 | 5.8136 |
| 5 | 0.0797951 | 0.826255 | 12.2633 |

| i | zo | Δz | z | xz |
|---|---|---|---|---|
| 1 | 0.0281358 | 0.0145597 | 0.0324661 | 0.0011854 |
| 2 | 0.0899415 | 0.216473 | 0.154324 | 1.06307 |
| 3 | 2.45764 | -2.29239 | 1.77584 | 3.35036 |
| 4 | 0.929686 | -0.934819 | 0.651655 | 3.78847 |
| 5 | 0.466148 | -1.55949 | 0.00233074 | 0.0285825 |

| j | yo | Δy | y |
|---|---|---|---|
| 1 | 2.45764 | -2.29239 | 1.77584 |
| 2 | 0.929686 | -0.934819 | 0.651655 |
| 3 | -0.466148 | 1.55949 | -0.00233074 |

```
Stepsize: (primal) 14.7454
          (dual) 0.297417

updated mu= 0.0823166
gap= 8.23166
```

## Iteration # 4

cx = 34.6618, by = 26.4301, +/xz = 8.23166, gap = 8.23166

Infeasibilities:

primal (dp): 4.09543E-10 3.62401E-10 1.23953E-9

dual (dd): 8.88178E-16 0 0 0 0

| i | xo | Δx | x |
|---|---|---|---|
| 1 | 0.0365118 | 2.36805 | 2.14121 |
| 2 | 6.88855 | -4.4382 | 2.94393 |
| 3 | 1.88663 | -1.77223 | 0.311487 |
| 4 | 5.8136 | -6.50834 | 0.029068 |
| 5 | 12.2633 | 5.91252 | 17.5182 |

| i | zo | Δz | z | xz |
|---|---|---|---|---|
| 1 | 0.0324661 | 0.116391 | 0.448625 | 0.9606 |
| 2 | 0.154324 | -0.0429454 | 0.000771622 | 0.0022716 |
| 3 | 1.77584 | -0.0640448 | 1.54685 | 0.481824 |
| 4 | 0.651655 | 0.0920335 | 0.980724 | 0.0285077 |
| 5 | 0.00233074 | 0.00325799 | 0.0139798 | 0.244901 |

| j | yo | Δy | y |
|---|---|---|---|
| 1 | 1.77584 | -0.0640448 | 1.54685 |
| 2 | 0.651655 | 0.0920335 | 0.980724 |
| 3 | -0.00233074 | -0.00325799 | -0.0139798 |

```
Stepsize: (primal) 0.888788
          (dual) 3.57553

updated mu= 0.017181
gap= 1.7181
```

## Iteration # 5

cx = 27.5669, by = 25.8488, +/xz = 1.7181, gap = 1.7181

Infeasibilities:

primal (dp): 4.55458E-11 4.03055E-11 1.37852E-10

dual (dd): -1.77636E-15 0 0 0 0

| i | xo | Δx | x |
|---|---|---|---|
| 1 | 2.14121E0 | -1.66109E-1 | 1.98864E0 |
| 2 | 2.94393E0 | 8.04421E-2 | 3.01781E0 |
| 3 | 3.11487E-1 | -3.37443E-1 | 1.55744E-3 |
| 4 | 2.9068E-2 | -5.22472E-3 | 2.42693E-2 |
| 5 | 1.75182E1 | 5.08776E-1 | 1.79855E1 |

| i | zo | Δz | z | xz |
|---|---|---|---|---|
| 1 | 4.48625E-1 | -4.05798E-1 | 2.75444E-2 | 5.4776E-2 |
| 2 | 7.71622E-4 | 5.04339E-3 | 6.00495E-3 | 1.81218E-2 |
| 3 | 1.54685E0 | 1.84052E-1 | 1.73783E0 | 2.70657E-3 |
| 4 | 9.80724E-1 | -2.13384E-1 | 7.59303E-1 | 1.84278E-2 |
| 5 | 1.39798E-2 | -1.3405E-2 | 6.98989E-5 | 1.25717E-3 |

| j | yo | Δy | y |
|---|---|---|---|
| 1 | 1.54685 | 0.184052 | 1.73783 |
| 2 | 0.980724 | -0.213384 | 0.759303 |
| 3 | -0.0139798 | 0.013405 | -0.0000698989 |

```
Stepsize: (primal) 0.918467
          (dual) 1.03766

updated mu= 0.000952893
gap= 0.0952893
```

## Iteration # 6

cx = 27.0209, by = 26.9256, +/xz = 0.0952893, gap = 0.0952893

Infeasibilities:

primal (dp): 3.96128E-12 3.10774E-12 1.09495E-11

dual (dd): -8.88178E-16 0 0 0 0

| i | xo | Δx | x |
|---|---|---|---|
| 1 | 1.98864E0 | 1.08473E-2 | 2.00018E0 |
| 2 | 3.01781E0 | -1.6781E-2 | 2.99997E0 |
| 3 | 1.55744E-3 | -1.02002E-3 | 4.73056E-4 |
| 4 | 2.42693E-2 | -2.27147E-2 | 1.21346E-4 |
| 5 | 1.79855E1 | 1.28874E-2 | 1.79992E1 |

| i | zo |
|---|---|
| 1 | 2.75444E-2 |
| 2 | 6.00495E-3 |
| 3 | 1.73783E0 |
| 4 | 7.59303E-1 |
| 5 | 6.98989E-5 |

| j | yo | Δy | y |
|---|---|---|---|
| 1 | 1.73783 | 0.0121683 | 1.75009 |
| 2 | 0.759303 | -0.00937437 | 0.749863 |
| 3 | -0.0000698989 | 0.0000169679 | -0.0000528118 |

```
Stepsize: (primal) 1.0631
          (dual) 1.00703

updated mu= 0.0000307315
gap= 0.00307315
```

## Iteration # 7

cx = 27.0009, by = 26.9978, +/xz = 0.00307315, gap = 0.00307315

Infeasibilities:

primal (dp): -2.46914E-13 -1.88294E-13 -6.89226E-13

dual (dd): 0 8.88178E-16 0 0 0

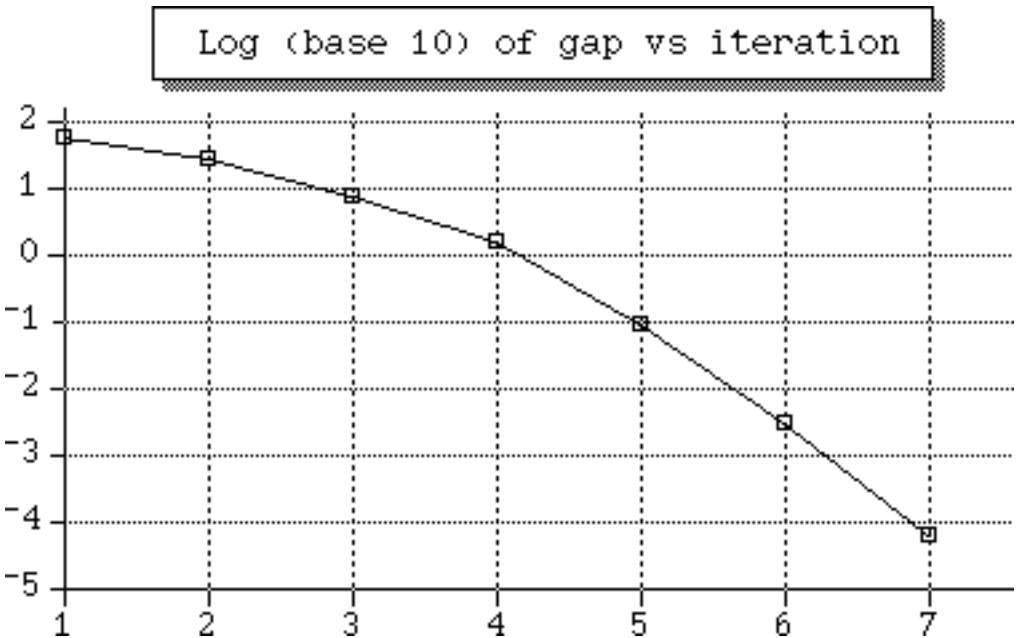
| i | xo | Δx | x |
|---|---|---|---|
| 1 | 2.00018E0 | -1.87543E-4 | 1.99998E0 |
| 2 | 2.99997E0 | 5.35786E-5 | 3.00003E0 |
| 3 | 4.73056E-4 | -4.55472E-4 | 2.36528E-6 |
| 4 | 1.21346E-4 | -8.03859E-5 | 3.82746E-5 |
| 5 | 1.79992E1 | 7.23401E-4 | 1.8E1 |

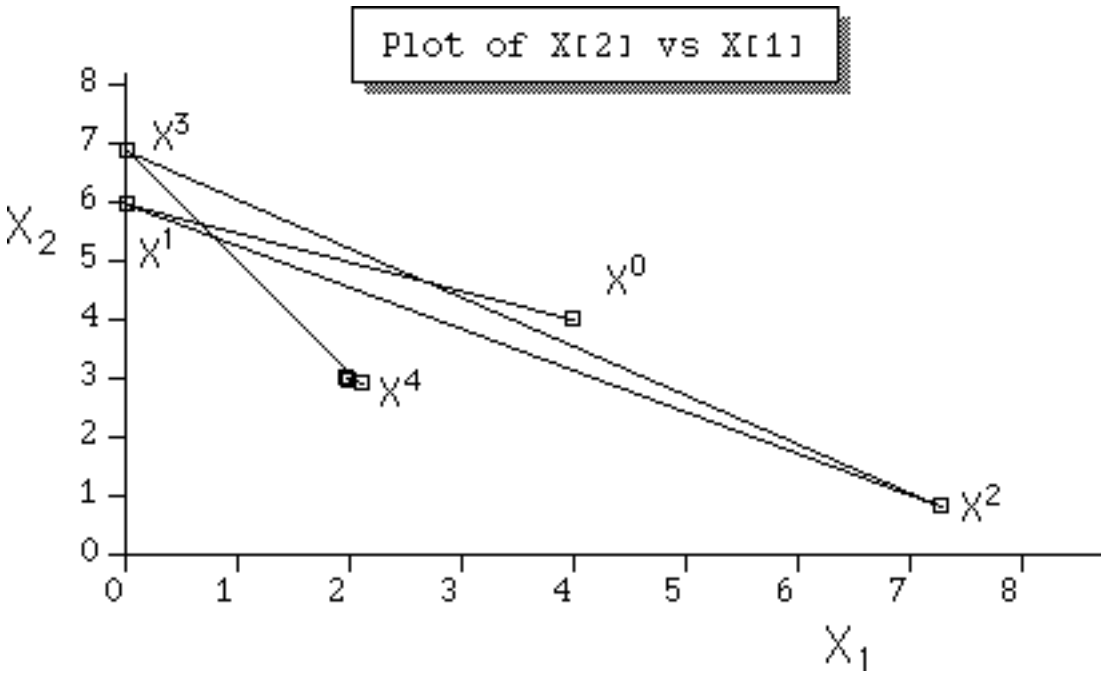
| i | zo | Δz | z | xz |
|---|---|---|---|---|
| 1 | 1.37722E-4 | -1.22345E-4 | 1.19272E-5 | 2.38542E-5 |
| 2 | 3.09413E-4 | -2.99175E-4 | 1.80171E-6 | 5.40519E-6 |
| 3 | 1.75009E0 | -9.02812E-5 | 1.74999E0 | 4.13923E-6 |
| 4 | 7.49863E-1 | 1.37655E-4 | 7.50005E-1 | 2.87061E-5 |
| 5 | 5.28118E-5 | -5.11065E-5 | 2.64059E-7 | 4.75306E-6 |

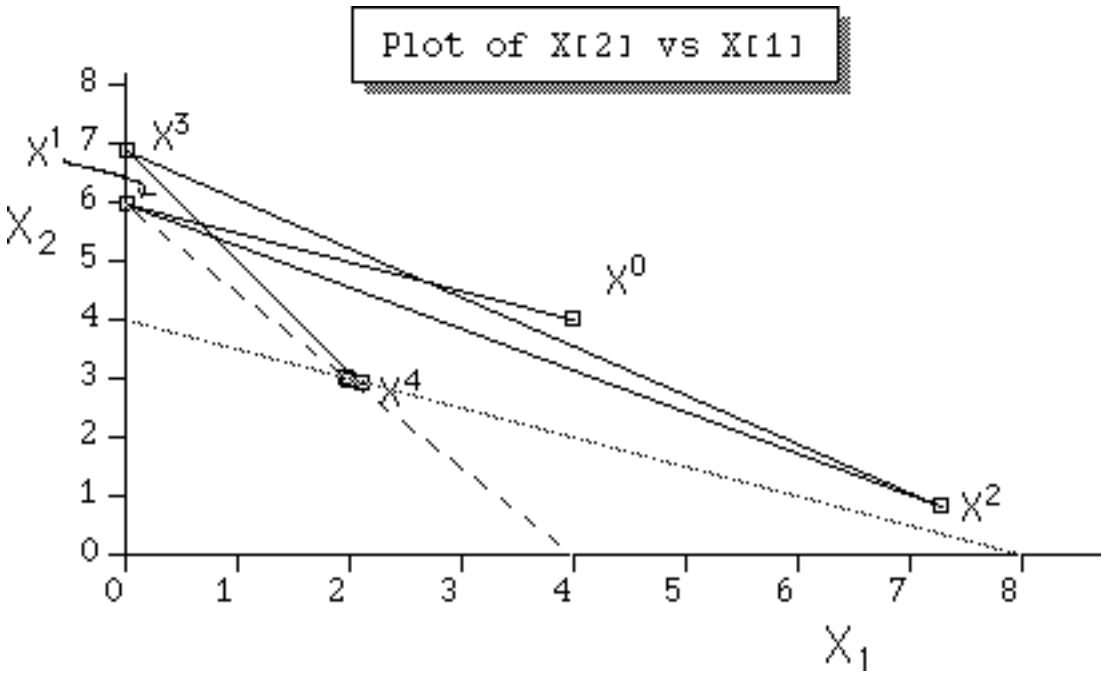
| j | yo | Δy | y |
|---|---|---|---|
| 1 | 1.75009E0 | -9.02812E-5 | 1.74999E0 |
| 2 | 7.49863E-1 | 1.37655E-4 | 7.50005E-1 |
| 3 | -5.28118E-5 | 5.11065E-5 | -2.64059E-7 |

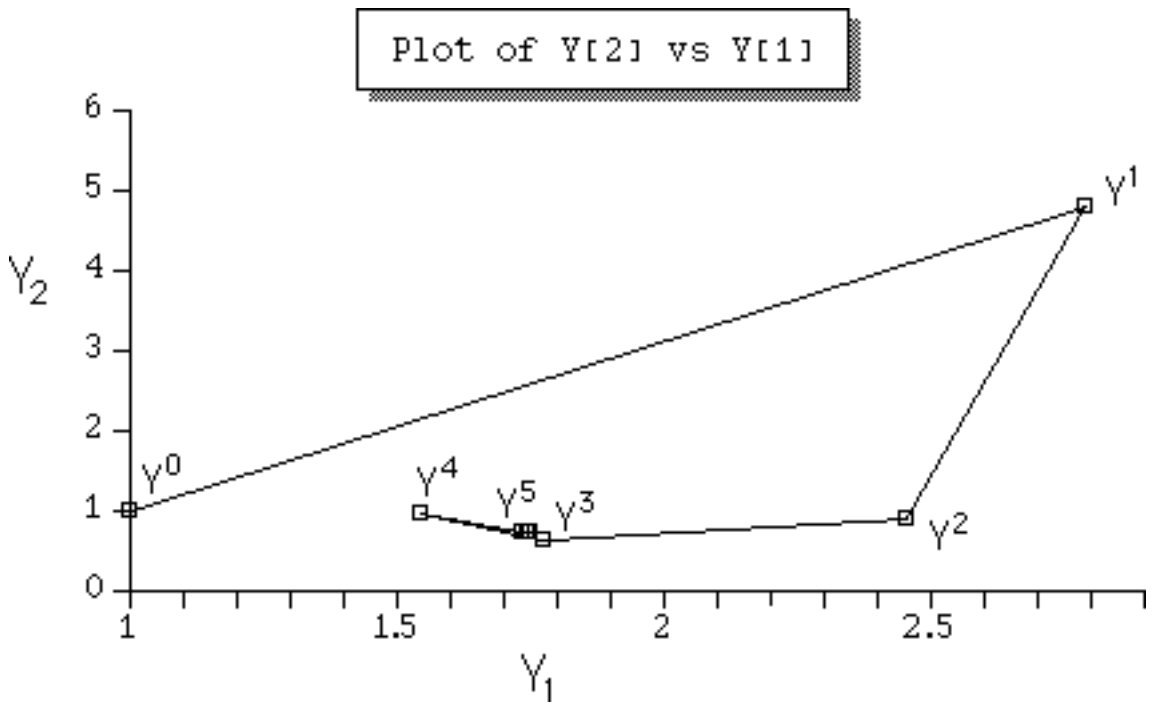
Stepsize: (primal) 1.03341
(dual) 1.0282

updated mu= 6.68578E-7
gap= 0.0000668578


Log (base 10) of gap vs iteration
2
1
0
-1
-2
-3
-4
-5
1
2
3
4
5
6
7


Plot of X[2] vs X[1]
X2
X1
X3
X1
X0
X4
X2
0
1
2
3
4
5
6
7
8


Plot of X[2] vs X[1]
X1
X2
X0
X1
X2
X3
X4
0
1
2
3
4
5
6
7
8


Plot of Y[2] vs Y[1]
6
5
4
3
2
1
0
$Y_2$
1
1.5
2
2.5
$Y_1$
$Y^0$
$Y^1$
$Y^2$
$Y^3$
$Y^4$
$Y^5$