*(also known as the "**Christmas Tree Problem**")*

A **one-stage** stochastic inventory replenishment problem

characterized by

- a ***single opportunity to order*** the commodity before demand occurs
- inventory remaining after demand occurs is ***obsolete***

Consider a problem with

a ***single commodity*** and

a single opportunity to replenish the inventory:

Notation:

- Current inventory level is ***s***.
- You must choose the amount ***z*** of commodity to add to the inventory, which will be delivered instantaneously.
- After replenishment, a demand for ***D*** units (a random variable) of the commodity will occur.
- Selling price is denoted by ***r***, and the purchase cost is ***c*** (< ***r***). A salvage value ***v*** (≤ c) is received for any inventory remaining after demand has occurred.

Further notation:

- a = s + z = amount available to meet demand
- minimum { a, D} = sales
- $(a-D)^+ \equiv \max\{0, a-D\}$ = residual stock after demand occurs
- $(D-a)^+ \equiv \max\{0, D-a\}$= sales lost to excess demand
- net revenue =

$$B(a) = r\left[a-(a-D)^+\right] - cz + v(a-D)^+$$

$$= (r-c)a + cs - (r-v)(a-D)^+$$

Revenue is a random variable, with expected value

$$E\{B(a)\} = (r-c)a + cs - (r-v)E\{(a-D)^+\}$$

$$= (r-c)a + cs - (r-v)\int_0^a (a-x)\,dF(x)$$

Example: suppose that D is uniformly distributed over the interval $[\mu-\delta, \mu+\delta]$ where $0 < \delta < \mu$.

Then

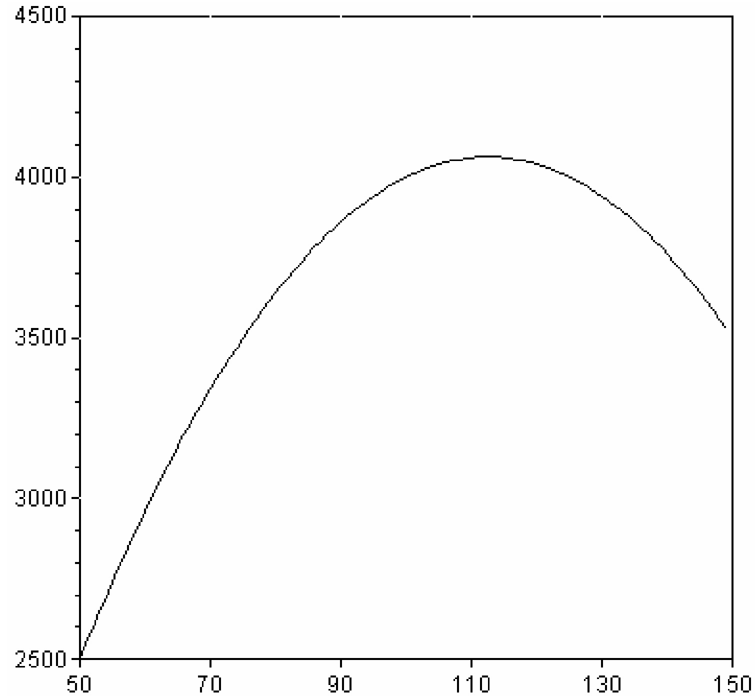
$$E\left\{(a-D)^{+}\right\} = \int_0^a (a-x)\,dF(x)$$

$$= \begin{cases} 0 & \text{if } a \le \mu-\delta \\ \dfrac{1}{2\delta}\displaystyle\int_{\mu-\delta}^{a}(a-x)\,dx = \dfrac{(a-\mu+\delta)^2}{4\delta} & \text{if } \mu-\delta < a \le \mu+\delta \\ a-\mu & \text{if } a > \mu+\delta \end{cases}$$

Then, denoting the expected benefit by $\Phi(s,a)$, we have

$$\Phi(s,a) = (r-c)a + cs - (r-v)\begin{cases} 0 & \text{if } a \le \mu - \delta \\ \dfrac{(a-\mu+\delta)^2}{4\delta} & \text{if } \mu - \delta < a \le \mu + \delta \\ a - \mu & \text{if } \mu + \delta < a \end{cases}$$

Plot of $\Phi(0,a)$ with selling price r=100, purchase cost = c = 50, salvage value v = 20, and D uniform in [50, 150] :



Within the interval $[\mu \pm \delta]$ the function $\Phi(0,a)$ has first derivative

$$\frac{\partial}{\partial a}\Phi(0,a) = r - c - 2(r-v)\frac{a-\mu+\delta}{4\delta}$$

and second derivative

$$\frac{\partial^2}{\partial a^2}\Phi(0,a) = -\frac{(r-v)}{2\delta} < 0$$

Therefore $\Phi(0,a)$ is a concave function, and simple calculus

shows that it has a maximum at

$$a^* = (\mu - \delta) + \frac{2\delta(r-c)}{r-v}$$

(so that, in particular, given r =100, c =50, v =20, μ =100, & δ =50

then the optimal inventory level is $a^* = 900/8$ = 112.5)

Value of Stochastic Solution (**VSS**):

If we were to have solved the problem of maximizing the benefit, *assuming that D assumes its expected value*, then clearly the optimal value *a** is the expected demand μ and the expected revenue using this replenishment level, assuming s<μ, is

$$\Phi(s,\mu) = (r-c)a + cs - (r-v)\frac{(a-\mu+\delta)^2}{4\delta}$$

Assuming the specified parameters, this expected revenue is $\Phi(0,100) = 4000$, while the maximum expected benefit (using non-integer replenishment value a*=112.5) is $\Phi(0,112.5) = 4062.50$. The Value of the Stochastic Solution is the difference,

$$\Phi(s,a^*) - \Phi(s,\mu) = 62.5.$$

In general, if the demand D has density function $f(x)$ and distribution function $F(x)$ with $F(0)=0$, then the expected revenue is

$$\Phi(a,s)=(r-c)a+cs-(r-v)\int_0^a (a-x)f(x)\,dx$$

In order to maximize this function with respect to the replenishment quantity *a*, then (since the upper limit of the integration is a function of *a*) we must use **Leibnitz' Rule** in order to find its derivative.

Leibnitz' Rule gives us the first derivative

$$\frac{d}{da}\Phi(0,a) = (r-c)-(r-v)\left[\int_0^a \frac{d}{da}(a-x)f(x)\,dx + (a-a)\frac{d}{da}a-(a-0)\frac{d}{da}0\right]$$
$$= (r-c)-(r-v)F(a)$$

Setting this derivative equal to zero yields the stationary point at the value a such that

$$F(a) = \frac{r-c}{r-v},$$

That is, assuming that a^* is not required to assume integer or discrete values,

the optimal replenishment quantity is

$$a^* = F^{-1}\left(\frac{r-c}{r-v}\right)$$

Two-stage Stochastic Linear Programming with Recourse

The newsboy problem can also be formulated as a 2-stage stochastic LP with

- first-stage variable

  x = the replenishment quantity

- second-stage (*recourse*) variables

  y_1 = quantity sold

  and

  y_2 = quantity salvaged after demand occurs

The 2-stage stochastic LP problem is

$$\textit{Maximize} \quad -cx + E_D Q(x, D)$$

where

$$Q(x, D) = \max_y \quad ry_1 + vy_2$$

$$\text{subject to } \; y_1 + y_2 \le x,$$

$$0 \le y_1 \le D, \quad 0 \le y_2$$

This is a problem with ***simple recourse***: the solution of the second-stage problem can be written in closed form as

$$y_1 = \min\{x, D\} \quad \& \quad y_2 = \max\{x - D, 0\}$$

It is interesting to note that the form of the optimal solution to the newsboy problem is that of a

Chance-constrained Linear Program:

$$\textit{Minimize } x$$

$$P\{x \geq D\} \geq \alpha = \frac{r-c}{r-v}$$

since

$$P\{x \geq D\} \geq \alpha \quad \Leftrightarrow \quad F(x) \geq \alpha \quad \Leftrightarrow \quad x \geq F^{-1}(\alpha)$$