Multistage Stochastic LP

with Recourse

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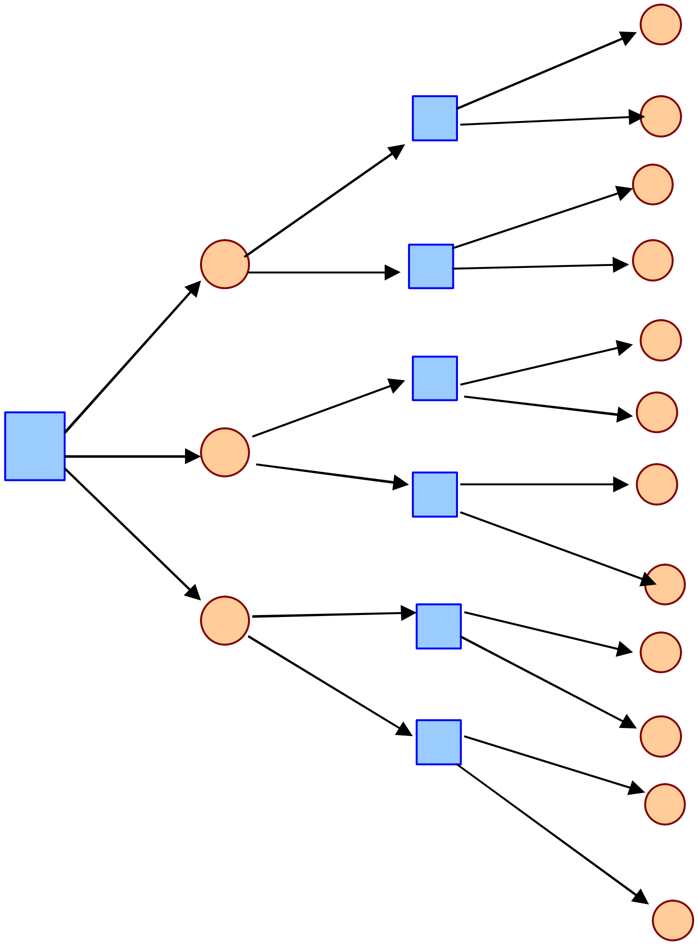
Assume:

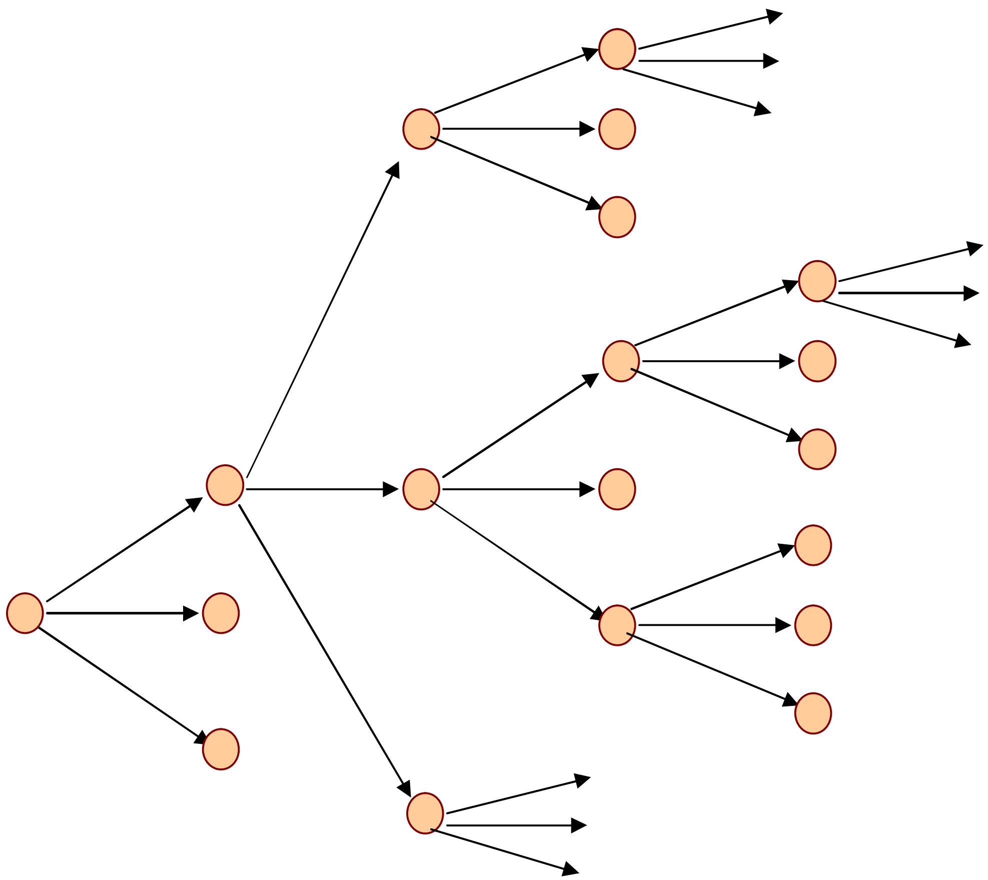
- $\omega = \left(\omega^1, \omega^2, \ldots \omega^H\right)$ is revealed at H different points in time (H is time horizon = # of stages)
- ω^t has discrete distribution $F_{\omega^t|\omega^{t-1}}$ which is conditional upon the previous outcome ω^{t-1}
- x^t (decisions at stage t) depend upon both previous decisions $\left(x^1, x^2, \ldots x^{t-1}\right)$ and previous outcomes $\left(\omega^1, \omega^2, \ldots \omega^t\right)$
- Recourse is *complete*, i.e., all optimization problems are feasible with respect to both random outcomes and previous decisions.

Decision Tree Representation (squares ~ decisions, circles ~ random outcomes)



Scenario Tree: Each node in the tree corresponds to a **scenario**.

Each stage t scenario *j* has a single **ancestor** scenario *a(j)* at stage t-1, and perhaps several **descendent** scenarios at stage t+1.



The Multistage Stochastic Linear Program (with H stages) can be stated:

$$\underset{x_1}{\text{Min}}\ c_1x_1 + E\left\{\min_{x_2}\ c_2\left(\omega^1\right)x_2\left(\omega^1\right) + \cdots + E\left[\min_{x_H}\ c_H\left(\omega^{H-1}\right)x_H\left(\omega^{H-1}\right)\right]\cdots\right\}$$

s.t.

$$W_1x_1 = h_1$$

$$T_1\left(\omega^1\right)x_1 + W_2x_2\left(\omega^1\right) = h_2\left(\omega^1\right)$$

$$\vdots$$

$$T_{H-1}\left(\omega^{H-1}\right)x_{H-1} + W_Hx_H\left(\omega^{H-1}\right) = h_H\left(\omega^{H-1}\right)$$

$$x_1 \geq 0, x_2\left(\omega^1\right) \geq 0, \cdots x_H\left(\omega^{H-1}\right) \geq 0$$

Recursive statement of problem:

$$\underset{x_1}{\text{Min}} \quad c_1 x_1 + Q_2(x_1)$$

subject to

$$W_1 x_1 = h_1, \quad x_1 \geq 0$$

where, for t=2, 3, ...H:

$$Q_t(x_t) = \sum_j p_t^j Q_t^j(x_t^j)$$

$$Q_t^j\left(x_{t-1}^{a(j)}\right) = \min_{x_t}\left\{c_t^j x_t + Q_{t+1}(x_t) : \; W_t x_t = h_t^j - T_{t-1}^{a(j)} x_{t-1}^{a(j)}, x_t \geq 0 \right\}$$

The function $Q_t^j(x)$ is convex and (in the case of *discrete* random outcomes) piecewise-linear.

Nested Benders' Decomposition

The piecewise-linear function $Q_t^j(x_t)$ in the problem at scenario *j* of stage *t* is approximated by a master problem

$$\underline{Q}_t^j(x_t) = \min \quad c_t^j x_t + \theta$$

subject to

$$W_t x_t = h_t^j - T_{t-1}^{a(j)} x_{t-1}^{a(j)}$$
$$\theta \geq \alpha_t^{j,k} x_t + \beta_t^{j,k}, \qquad k = 1,2,\ldots K_t^j$$
$$x_t \geq 0$$

where

$x_{t-1}^{a(j)}$ is the "trial" decision from the ancestor scenario *a(j)*, and

$\alpha_t^{j,k} x_t + \beta_t^{j,k}$ are the K_t^j supports of $E\left[Q_{t+1}^j(x_t)\right]$ generated by the descendents of scenario *j*.

Primal subproblem:

$$\underline{Q}_t^j(x_t) = \min \quad c_t^j x_t + \theta$$
subject to
$$W_t x_t = h_t^j - T_{t-1}^{a(j)} x_{t-1}^{a(j)}$$
$$\theta \geq \alpha_t^{j,k} x_t + \beta_t^{j,k}, \qquad k = 1,2,\ldots K_t^j$$
$$x_t \geq 0$$

Dual of subproblem:

$$\underline{Q}_t^j(x_t) = Max \quad \left[h_t^j - T_{t-1}^{a(j)} x_{t-1}^{a(j)} \right] u + \beta_t^j v$$

subject to
$$uW_t - vA_t^j \geq c_t^j$$
$$\sum_k v_k = 1$$
u unrestricted in sign;
$$v \geq 0$$

Dual LP has a feasible region *independent* of scenario and decision variable $x_{t-1}^{a(j)}$ from ancestor scenario, but *dependent* upon the optimality cuts $\left(A_t^j, \beta_t^j\right)$ which have been generated by the descendents.

Passing Optimality Cuts

If the dual solution $(\hat{u},\hat{v})$ is the k[th] extreme point to be identified, then the optimality cut which is passed up the scenario tree to be added to the ancestor problem is

$$\left[h_t^j - T_{t-1}^{a(j)} x_{t-1}^{a(j)}\right]\hat{u} + \beta_t^j \hat{v} \equiv$$

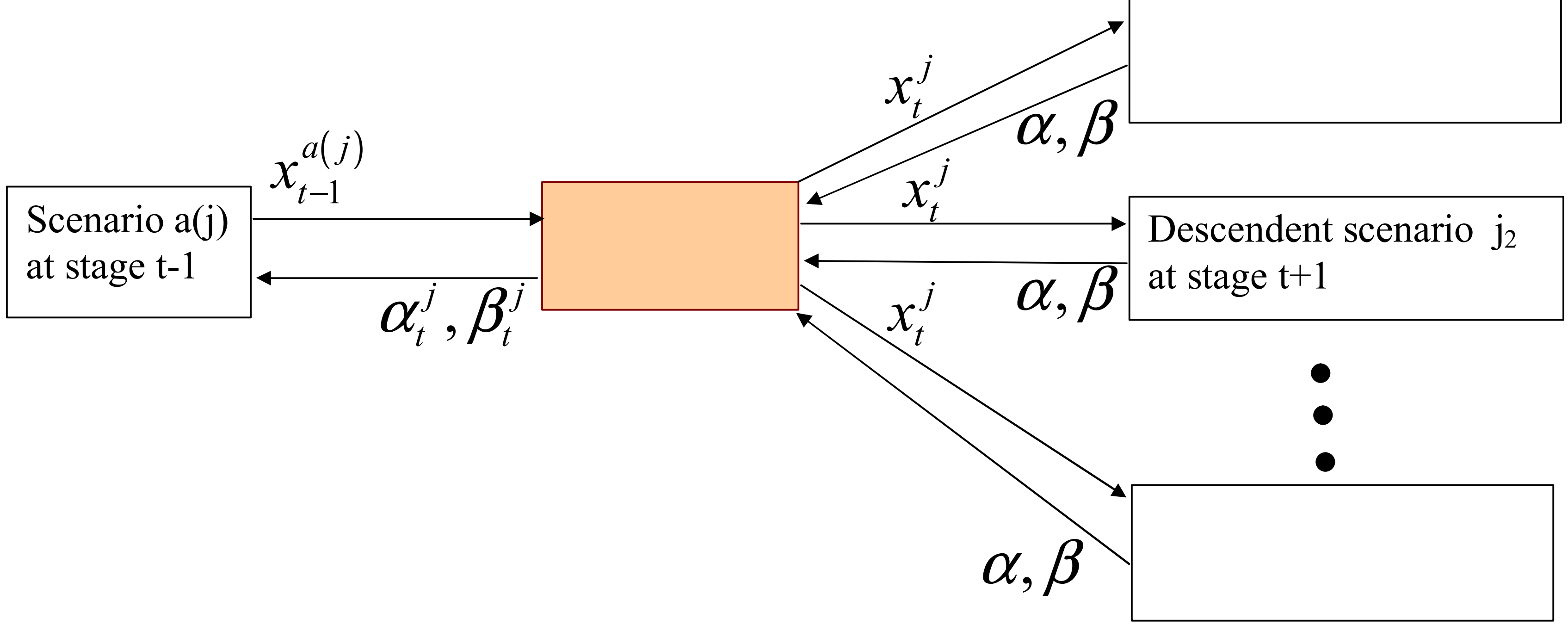
$$\alpha_{t-1}^{a(j),k} x_{t-1}^{a(j)} + \beta_{t-1}^{a(j),k}$$

where

$$\alpha_{t-1}^{a(j),k} = -\hat{u} T_{t-1}^{a(j)}, \quad \beta_{t-1}^{a(j),k} = h_t^j \hat{u} + \beta_t^j \hat{v}$$

After solving each approximating problem above,

- the optimality cuts (or dual variables $(\hat{u}, \hat{v})$) are passed up to the ancestor scenario, and
- the primal variables x_t^j are passed down to the descendent scenarios.



When computations are not done in parallel, there are many possible sequences in which the problems may be solved….

most common is "Fast-Forward, Fast-Backward"   (reversing direction only when reaching the bottom or top of the scenario tree).