Multiple-Facility Location in the Plane

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Given

a set of demand points (X_j, Y_j), $j=1,2,\ldots n$
with volume of demand w_j

Find

a set of m facility locations, and
an allocation of the demand points to
these facilities, so as to minimize
the sum of the transportation costs

Two aspects of decisions to be made:

Location

$(\hat{X}^i, \hat{Y}^i)$ = location of facility # i

Allocation

J_i = set of demand points to be served by facility #i

where

$J_1 \cup J_2 \cup \cdots \cup J_m = \{1, 2, \ldots n\}$

Objective is a function of both the location and allocation, i.e., of $(\hat{X}^i, \hat{Y}^i)$ & J_i

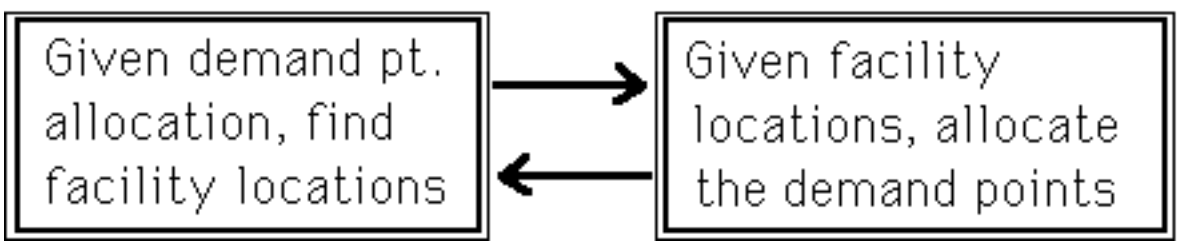
Total Transportation Cost

$$C(\hat{X}^1, \hat{Y}^1, J_1, \ldots \hat{X}^m, \hat{Y}^m, J_m) =$$

$$\sum_{i=1}^{m} \sum_{j \in J_i} w_j \sqrt{(\hat{X}^i - X_j)^2 + (\hat{Y}^i - Y_j)^2}$$

Optimizing simultaneously with respect to
the facility locations
the allocation of demand points to facilities
is quite difficult.

One heuristic approach is to alternate between a location problem & allocation problem:



Heuristic Algorithm

1. Partition the set of n demand points into m subsets J_i, $i=1,\ldots m$, $J_1 \cup \ldots \cup J_m = \{1,2,\ldots n\}$

2. ***Location:*** For each subset J_i, solve optimally a single-source facility location problem, to obtain facility locations $(\hat{X}^i, \hat{Y}^i)$, $i=1,2,\ldots n$

3. ***Re-Allocation:*** Allocate each demand point to the nearest facility location. That is,

   If $j \in J_{i'}$ but $d_j(\widehat{X}^{i''}, \widehat{Y}^{i''}) < d_j(\widehat{X}^{i'}, \widehat{Y}^{i'})$

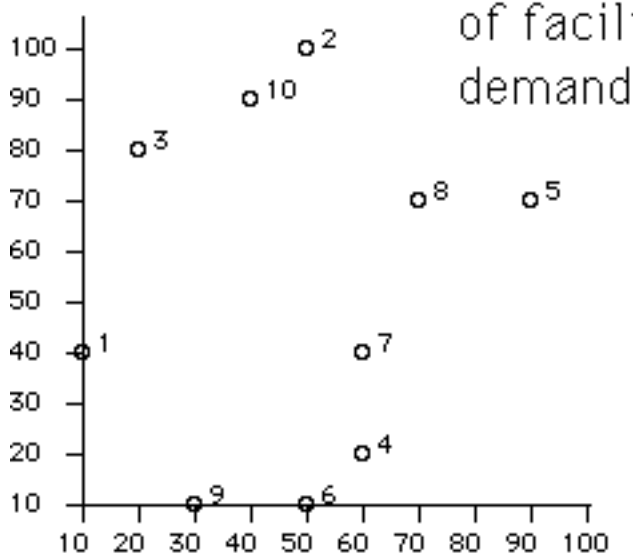
   then $J_{i''} = J_{i''} \cup \{j\}$, $J_{i'} = J_{i'} / \{j\}$

4. If no demand point was re-allocated in step 3, then STOP. Otherwise, return to step 2.

Example

Cost of operating facility = $75/day

Find the optimal number & locations of facilities to satisfy the daily demand:



i	W_i
1	0.75
2	0.6
3	0.75
4	0.6
5	1.2
6	0.5
7	0.4
8	0.5
9	1.05
10	0.6

shipping cost is W_i times the distance shipped

Single-Source Weber Problem

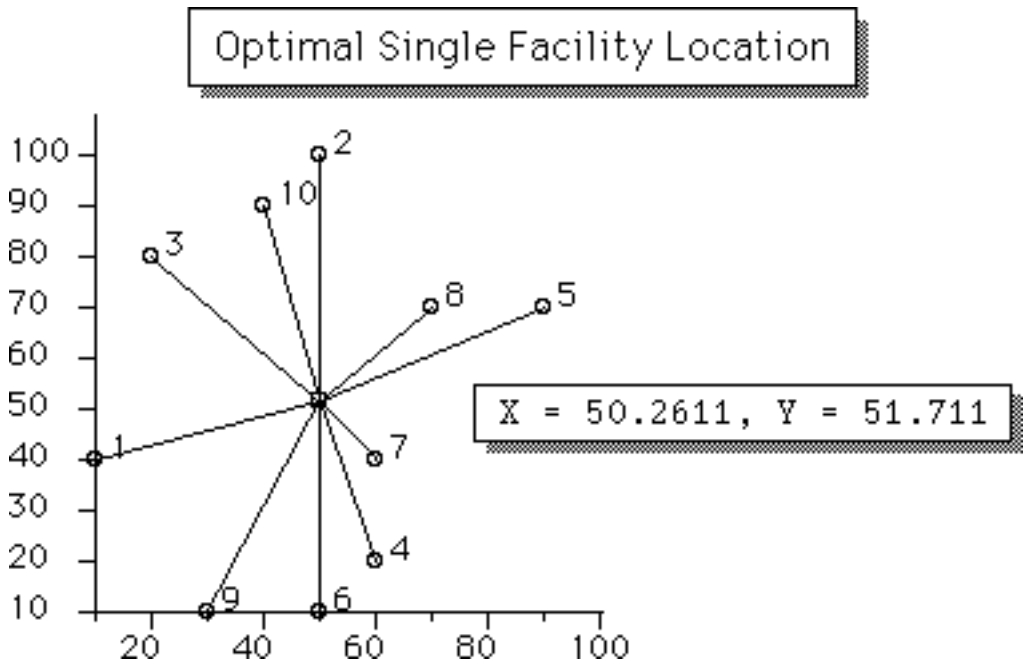
The optimal single source serving the demand points is located at:

X = 50.2611, Y = 51.711

i	C[i]
1	31.4479
2	28.964
3	31.058
4	19.9137
5	52.4907
6	20.864
7	6.09857
8	13.4506
9	48.7037
10	23.774

Total daily cost:
276.765+75
= 351.765

Total cost = 276.765
(shipping cost.)



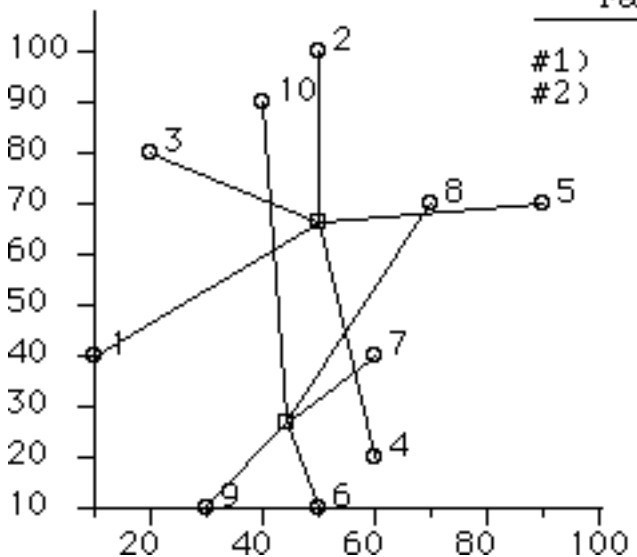
Locating Two Facilities

Let's begin by partitioning the set of demand points into two subsets:

$$J_1 = \{1,2,3,4,5\}$$
$$J_2 = \{6,7,8,9,10\}$$

We then find the optimal location of a facility to serve the demand points J_1 , and another facility to serve the demand points J_2.

Iteration 1



Facility Location			Serving Demand Pts
#1)	50.3213	66.6798	1 2 3 4 5
#2)	44.5305	27.3635	6 7 8 9 10

Facility locations:

#1: (50.32, 66.68)
#2: (44.51, 27.33)

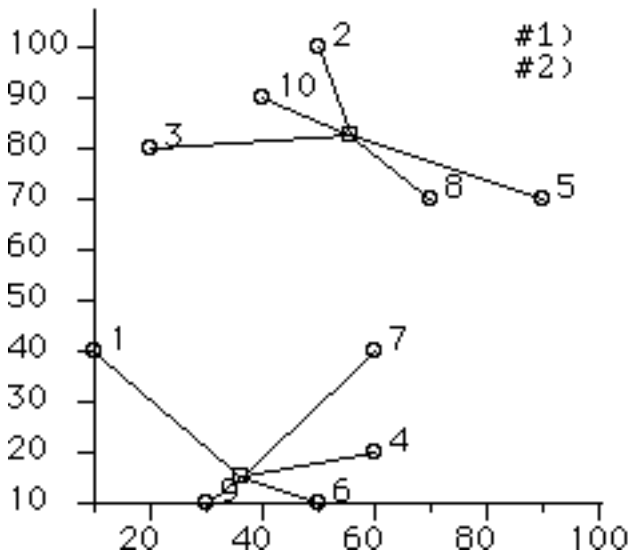
Allocation of Demand Points

i	Shipping Cost		Old Source		Source
1	36.2617	27.5775	1	→	2
2	19.993	43.7053	1		1
3	24.8386	43.5539	1		1
4	28.6036	10.2796	1	→	2
5	47.7808	74.7991	1		1
6	28.3404	9.10228	2		2
7	11.3525	7.98987	2		2
8	9.97841	24.8323	2	→	1
9	63.2233	23.7733	2		2
10	15.3013	37.6801	2	→	1

Total shipping cost after re-allocating demand: 196.615
Total shipping cost before re-allocation: 260.856
(Improvement from re-allocation: 64.2409)

Iteration 2

Facility	Location		Serving Demand Pts
#1)	56.1573	82.6547	2 3 5 8 10
#2)	36.4130	15.7924	1 4 6 7 9



Facility locations:

#1: (56.10, 82.69)
#2: (36.38, 15.78)

Allocation of Demand Points

i	Shipping Cost		Old Source	Source
1	47.1363	26.8711	2	2
2	11.0435	51.178	1	1
3	27.191	49.7041	1	1
4	37.6634	14.3756	2	2
5	43.3575	91.4683	1	1
6	36.4576	7.38512	2	2
7	17.131	13.5195	2	2
8	9.37765	31.8848	1	1
9	81.0808	9.07374	2	2
10	10.6491	44.5765	1	1

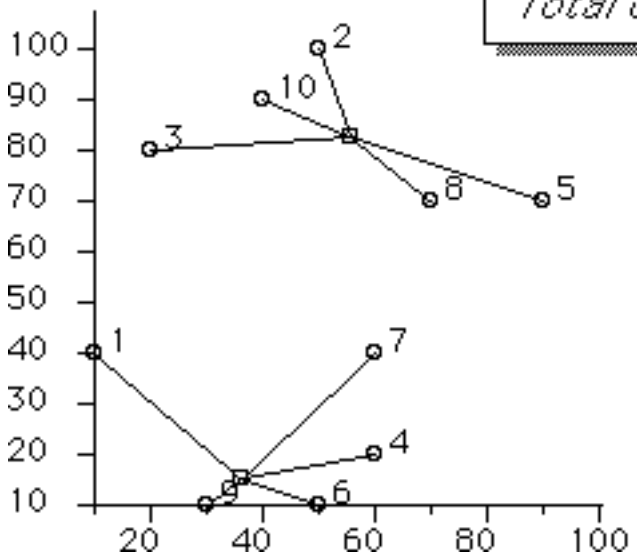
Total shipping cost after re-allocating demand: 172.844

Total shipping cost before re-allocation: 172.844

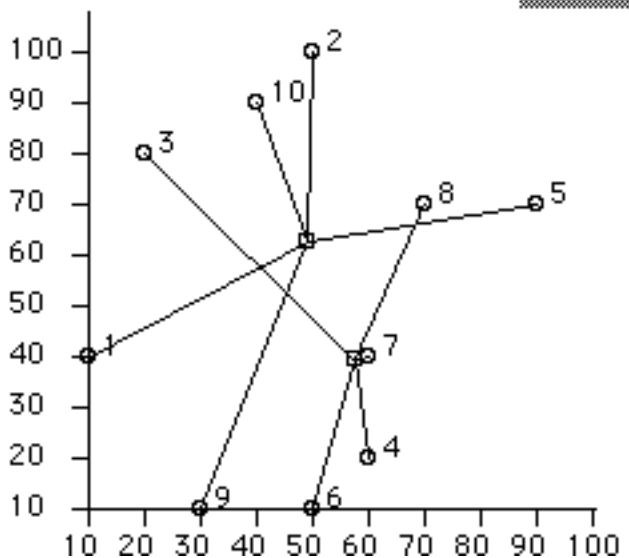
(Improvement from re-allocation: 0)

Converged!

Shipping cost:	172.84
Operating cost:	2x75
Total daily cost:	322.84



Starting from another initial (random) allocation:



Iteration 1

	Facility Location		Serving Demand Pts
#1)	57.8549	39.6230	3 4 6 7 8
#2)	49.1617	63.0588	1 2 5 9 10

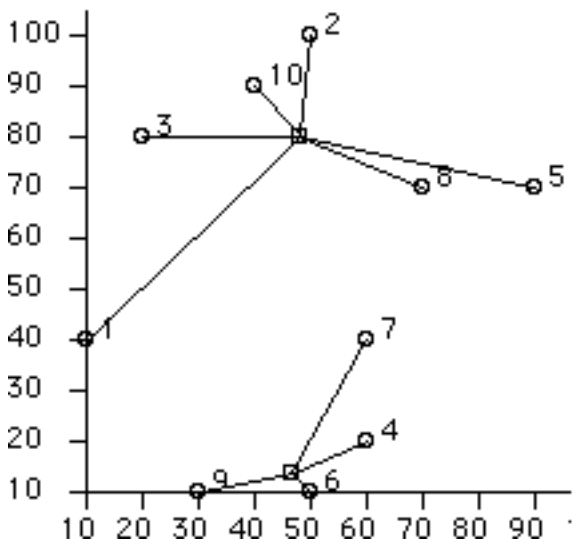
Facility locations:

#1: (57.85, 39.62)
#2: (49.16, 63.06)

Allocation of Demand Points

i	Shipping Cost		Old Source	Source
1	35.8923	34.0846	2	2
2	36.5315	22.1704	2	2
3	41.5103	25.2941	1	2
4	11.8439	26.6412	1	1
5	53.073	49.7088	2	2
6	15.3233	26.5327	1	1
7	0.87121	10.1916	1	1
8	16.3575	10.982	1	2
9	42.6953	59.2335	2	1
10	32.0685	17.0738	2	2

Total shipping cost after re-allocating demand: 230.047
Total shipping cost before re-allocation: 268.177
(Improvement from re-allocation: 38.1299)



Iteration 2

	Facility Location		Serving Demand Pts
#1)	47.0367	13.7634	4 6 7 9
#2)	48.5206	80.2241	1 2 3 5 8 10

Facility locations:

#1: (47.04, 13.76)
#2: (48.52, 80.22)

Allocation of Demand Points

i	Shipping Cost		Old Source	Source
1	34.041	41.7704	2	1
2	51.7725	11.8987	2	2
3	53.6565	21.3911	2	2
4	8.63128	36.785	1	1
5	84.924	51.2651	2	2
6	2.39502	35.1198	1	1
7	11.7058	16.732	1	1
8	30.3721	11.8943	2	2
9	18.3198	76.2565	1	1
10	45.9364	7.7808	2	2

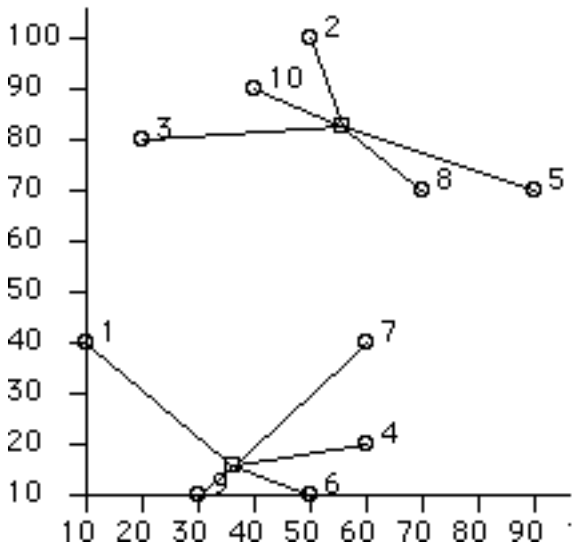
Total shipping cost after re-allocating demand: 179.323

Total shipping cost before re-allocation: 187.052

(Improvement from re-allocation: 7.72937)

Iteration 3

	Facility Location		Serving Demand Pts
#1)	36.4130	15.7924	1 4 6 7 9
#2)	56.1573	82.6547	2 3 5 8 10



Facility locations:

#1: (36.41, 15.79)
#2: (56.16, 82.65)

Allocation of Demand Points

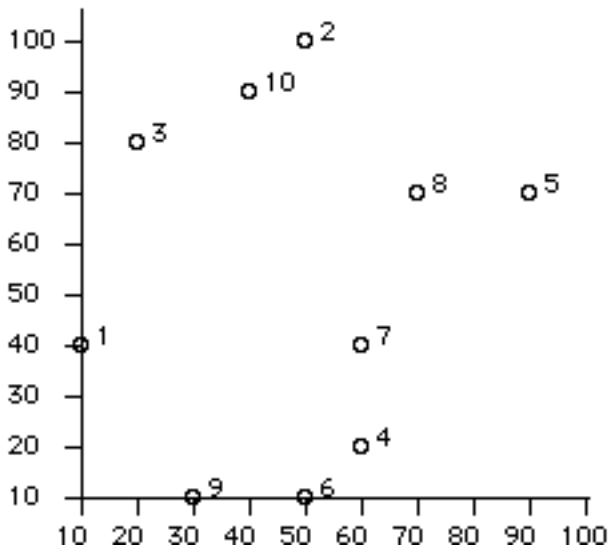
i	Shipping Cost		Old Source	Source
1	26.8711	47.1363	1	1
2	51.178	11.0435	2	2
3	49.7041	27.191	2	2
4	14.3756	37.6634	1	1
5	91.4683	43.3575	2	2
6	7.38512	36.4576	1	1
7	13.5195	17.131	1	1
8	31.8848	9.37765	2	2
9	9.07374	81.0808	1	1
10	44.5765	10.6491	2	2

Total shipping cost after re-allocating demand: 172.844
Total shipping cost before re-allocation: 172.844
(Improvement from re-allocation: 0)

Converged!

(same as earlier solution!)

Next we will find a solution with THREE facility locations.

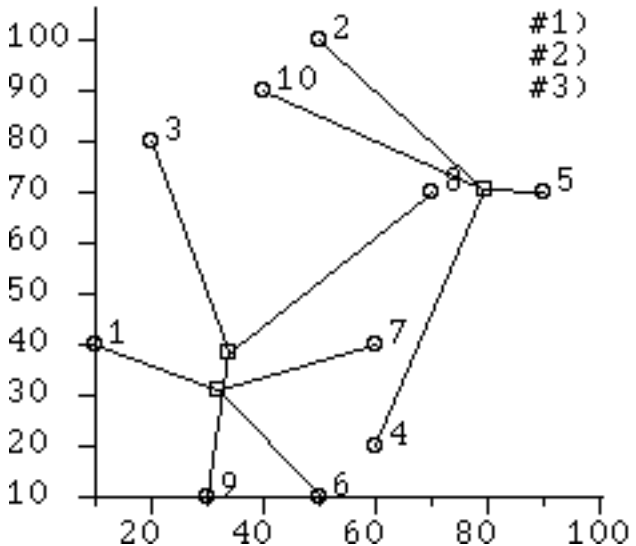


Random initial allocation:

Demand Pts
3 8 9
1 6 7
2 4 5 10

Iteration 1

	Facility Location		Serving Demand Pts
#1)	33.8389	38.5597	3 8 9
#2)	32.3758	31.4353	1 6 7
#3)	79.9591	70.9838	2 4 5 10



Facility locations:

#1: (33.84, 38.56)
#2: (32.38, 31.44)
#3: (79.96, 70.98)

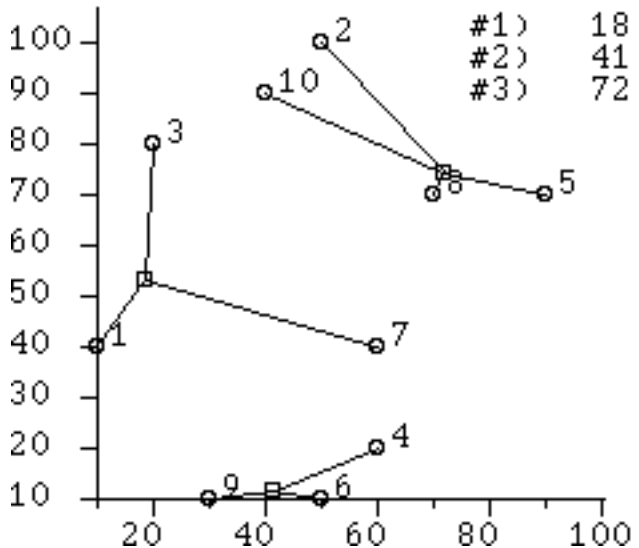
Allocation of Demand Points

i	Shipping Cost			Old Source		Source
1	17.9118	17.9692	57.3849	2	→	1
2	38.1181	42.4761	25.0243	3		3
3	32.7675	37.5876	45.4749	1		1
4	19.2456	17.9385	32.8508	3	→	2
5	77.2353	83.2058	12.1068	3		3
6	16.4076	13.8752	33.9727	2		2
7	10.4803	11.5686	14.7424	2	→	1
8	23.9589	26.9389	5.00379	1	→	3
9	30.2574	22.6449	82.7766	1	→	2
10	31.0848	35.4353	26.5519	3		3

Total shipping cost after re-allocating demand: 184.305
Total shipping cost before re-allocation: 226.931
(Improvement from re-allocation: 42.6256)

Iteration 2

	Facility Location		Serving Demand Pts
#1)	18.8634	53.4466	1 3 7
#2)	41.8018	11.6884	4 6 9
#3)	72.2591	74.2234	2 5 8 10



Facility locations:

#1: (18.86, 53.45)
#2: (41.80, 11.69)
#3: (72.26, 74.22)

Allocation of Demand Points

i	Shipping Cost			Old Source	Source
1	12.0787	31.9336	53.284	1	1
2	33.6038	53.2148	20.4344	3	3
3	19.9333	53.7797	39.433	1	1
4	31.8107	12.0038	33.3552	2	2
5	87.6446	90.7831	21.8841	3	3
6	26.7259	4.18512	33.9857	2	2
7	17.3114	13.4624	14.5411	1 →	2
8	26.8745	32.3859	2.39481	3	3
9	47.0937	12.5181	80.7236	2	2
10	25.3347	46.9994	21.5462	3	3

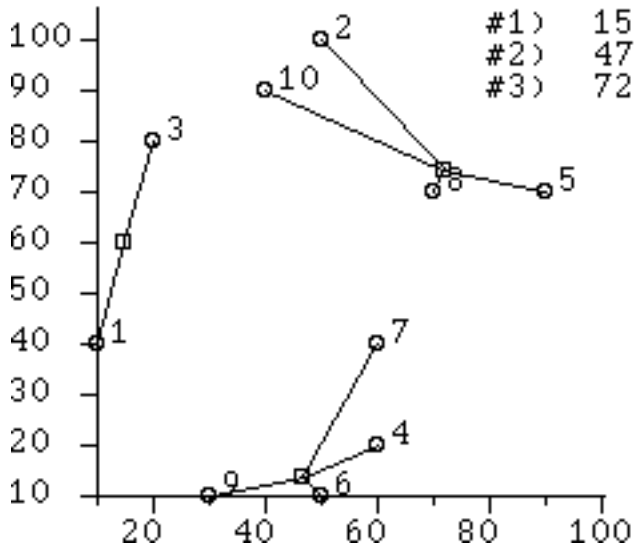
Total shipping cost after re-allocating demand: 140.441

Total shipping cost before re-allocation: 144.29

(Improvement from re-allocation: 3.84903)

Iteration 2

Facility Location			Serving Demand Pts
#1)	15.0000	60.0000	1 3
#2)	47.0367	13.7634	4 6 7 9
#3)	72.2591	74.2234	2 5 8 10



Facility locations:

#1: (15.00, 60.00)
#2: (47.04, 13.76)
#3: (72.26, 74.22)

Allocation of Demand Points

i	Shipping Cost			Old Source	Source
1	15.4616	34.041	53.284	1	1
2	31.8904	51.7725	20.4344	3	3
3	15.4616	53.6565	39.433	1	1
4	36.1248	8.63128	33.3552	2	2
5	90.7965	84.924	21.8841	3	3
6	30.5164	2.39502	33.9857	2	2
7	19.6977	11.7058	14.5411	2	2
8	27.9508	30.3721	2.39481	3	3
9	54.8116	18.3198	80.7236	2	2
10	23.4307	45.9364	21.5462	3	3

Total shipping cost after re-allocating demand: 138.235

Total shipping cost before re-allocation: 138.235

(Improvement from re-allocation: 0)

Converged!

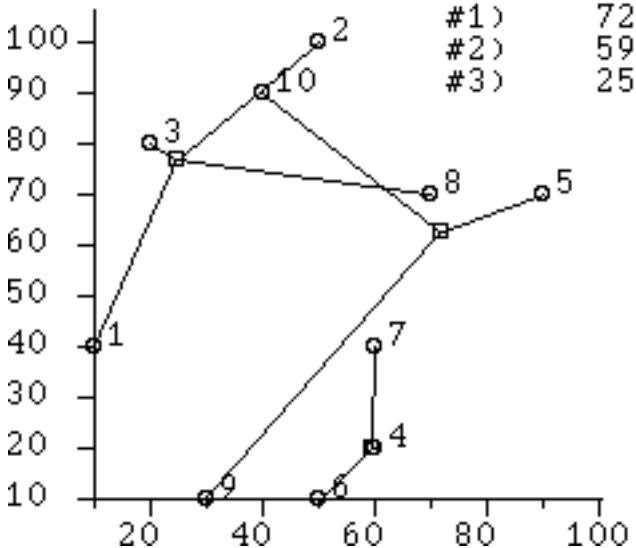
Starting from a different (random) allocation of the demand points:

Demand Pts
5 9 10
4 6 7
1 2 3 8

Will the algorithm converge to the same facility locations?

Iteration 1

	Facility Location		Serving Demand Pts
#1)	72.3572	63.0902	5 9 10
#2)	59.9781	20.0028	4 6 7
#3)	25.1492	77.1085	1 2 3 8



Facility locations:

#1: (72.36, 63.09)
#2: (59.98, 20.00)
#3: (25.15, 77.11)

Allocation of Demand Points

i	Shipping Cost			Old Source	Source
1	49.8712	40.3727	30.0612	3	3
2	25.8918	48.3702	20.2724	3	3
3	41.2651	54.0724	4.42914	3	3
4	26.8963	0.0132355	40.1416	2	2
5	22.7372	69.982	78.2871	1	1
6	28.8028	7.06434	35.781	2	2
7	10.4756	7.99887	20.3632	2	2
8	3.65039	25.4959	22.7053	3	→ 1
9	71.3127	33.1831	70.6478	1	→ 2
10	25.2508	43.6754	11.7994	1	→ 3

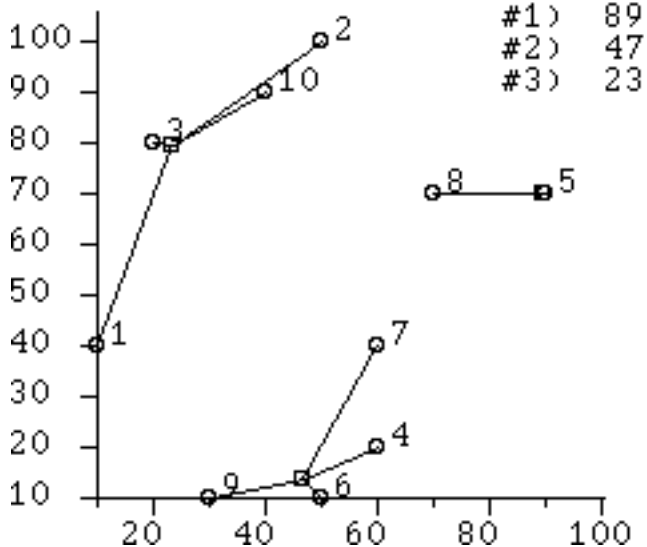
Total shipping cost after re-allocating demand: 141.209

Total shipping cost before re-allocation: 211.845

(Improvement from re-allocation: 70.636)

Iteration 2

	Facility Location		Serving Demand Pts
#1)	89.9924	70.0000	5 8
#2)	47.0367	13.7634	4 6 7 9
#3)	23.6396	79.8604	1 2 3 10



Facility locations:

#1: (89.99, 70.00)
#2: (47.04, 13.76)
#3: (23.64, 79.86)

Allocation of Demand Points

i	Shipping Cost			Old Source	Source
1	64.0747	34.041	31.5971	3	3
2	29.9964	51.7725	19.904	3	3
3	53.0274	53.6565	2.7317	3	3
4	34.9834	8.63128	42.0229	2	2
5	0.00908125	84.924	80.5068	1	1
6	36.0534	2.39502	37.3341	2	2
7	16.9684	11.7058	21.5812	2	2
8	9.99622	30.3721	23.6987	1	1
9	89.0898	18.3198	73.6568	2	2
10	32.3068	45.9364	11.5486	3	3

Total shipping cost after re-allocating demand: 116.839
Total shipping cost before re-allocation: 116.839
(Improvement from re-allocation: 0)

Converged!

Facility Location		Serving Demand Pts
#1) 15.0000	60.0000	1 3
#2) 47.0367	13.7634	4 6 7 9
#3) 72.2591	74.2234	2 5 8 10

Total Shipping Cost: 138.235

Facility Location		Serving Demand Pts
#1) 89.9924	70.0000	5 8
#2) 47.0367	13.7634	4 6 7 9
#3) 23.6396	79.8604	1 2 3 10

Total Shipping Cost: 116.839

Starting from two different initial allocations of the demand points, we obtain two different final solutions!

Using 3 facilities:

Shipping cost:	116.84
Operating cost:	3x 75.00
Total daily cost:	341.84

It appears that the optimal solution is to use 2 facilities:

Shipping cost:	172.84
Operating cost:	2x75
Total daily cost:	322.84