Some New Path-Following Algorithms for Convex Quadratic Programming

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stmt. of the QP problem

In path-following methods for convex quadratic programming, one must solve systems of equations of the form:

$$\begin{cases} Ax - y = b \\ -Qx + A^T w + s = c \\ XSe = \mu e \\ WYe = \mu e \end{cases}$$

This system consists of both linear and nonlinear equations, and are frequently solved using Newton's method.

The motivation for our current work was a presentation by Scott Burns (U. of Illinois) on the "Monomial Method" for solving certain systems of nonlinear equations.

Burns, Scott A., (1993). The Monomial Method and Asymptotic Properties of Algebraic Systems. University of Illinois.

__________ (1993). The Monomial Method: Extensions, Variations, and Performance Issues. University of Illinois.

__________ and A. Locascio (1991). "A Monomial-Based Method for Solving Systems of Non-Linear Algebraic Equations." Int'l J. for Numerical Methods in Engineering 31: 1295-1318.

- ☞ Arithmetic-Geometric Mean Inequality
- ☞ Condensation of Posynomials
- ☞ Posynomial Approximation of Signomials
- ☞ The "Monomial Method" for Solving Systems of Nonlinear Equations
- ☞ A "toy" LCP Example
- ☞ Application to Path-Following Algorithm
- ☞ Computational Experience

The Arithmetic-Geometric Mean Inequality

Simplest case: Given two positive numbers a&b,

their arithmetic mean $\frac{1}{2}a + \frac{1}{2}b$

is greater than or equal to their

geometric mean $\sqrt{a\,b}$

i.e., $\boxed{\frac{1}{2}a + \frac{1}{2}b \geq a^{\frac{1}{2}}\, b^{\frac{1}{2}}}$

with equality if & only if $a = b$

Arithmetic-Geometric Mean Inequality

$$\boxed{\tfrac{1}{2}\,\mathbf{a} + \tfrac{1}{2}\,\mathbf{b} \geq \mathbf{a}^{1/2}\,\mathbf{b}^{1/2}}$$

For example, let a=2 & b=8. Then this inequality is

$$5 = \underbrace{\tfrac{1}{2}\times 2 + \tfrac{1}{2}\times 8}_{\textit{Arithmetic mean}} \geq \underbrace{\sqrt{2\times 8} = 4}_{\textit{Geometric Mean}}$$

If a=4 & b=9,

$$6.5 = \underbrace{\tfrac{1}{2}\times 4 + \tfrac{1}{2}\times 9}_{\textit{Arithmetic mean}} \geq \underbrace{\sqrt{4\times 9} = 6}_{\textit{Geometric Mean}}$$

Arithmetic-Geometric Mean Inequality

$$\boxed{\tfrac{1}{2}\,a + \tfrac{1}{2}\,b \ \geq\ a^{1/2}\, b^{1/2}}$$

Proof: Let α & β be real numbers

and $a = \alpha^2 \geq 0$

$b = \beta^2 \geq 0$

Then $(\alpha - \beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2 \geq 0$

$$\Longrightarrow \quad \alpha^2 + \beta^2 \geq 2\alpha\beta$$

$$\Longrightarrow \quad \tfrac{1}{2}\,\alpha^2 + \tfrac{1}{2}\,\beta^2 \geq \alpha\,\beta \quad \Longrightarrow \quad \tfrac{1}{2}\,a + \tfrac{1}{2}\,b \geq \sqrt{a\,b}$$

The Arithmetic-Geometric Mean Inequality

The General Case: Let $x_1, x_2, \ldots x_n > 0$

and $\delta_1, \delta_2, \ldots \delta_n \geq 0$ and $\sum_{i=1}^{n} \delta_i = 1$

Then

$$\boxed{\sum_{i=1}^{n} \delta_i x_i \geq \prod_{i=1}^{n} x_i^{\delta_i}}$$

with equality if & only if $x_1 = x_2 = \ldots = x_n$

The Arithmetic-Geometric Mean Inequality

$$\sum_{i=1}^{n} \delta_i x_i \geq \prod_{i=1}^{n} x_i^{\delta_i}$$

If we let n=2, and $\delta_i = \frac{1}{2}$, then we obtain the earlier inequality,

$$\frac{1}{2} a + \frac{1}{2} b \geq a^{\frac{1}{2}} b^{\frac{1}{2}}$$

The Arithmetic-Geometric Mean Inequality

Writing $\mathbf{u}_i = \delta_i \mathbf{x}_i$, we get

Equivalent form:

$$\boxed{\sum_i \mathbf{u}_i \geq \prod_i \left({}^{\mathbf{u}_i}/_{\delta_i} \right)^{\delta_i}}$$

where $\delta_1, \delta_2, \cdots \delta_n \geq 0$ **and** $\sum_{i=1} \delta_i = 1$

with equality if & only if $\mathbf{u}_1/\delta_1 = \mathbf{u}_2/\delta_2 = \ldots = \mathbf{u}_n/\delta_n$

Condensation of Posynomials

$$g(x_1, x_2, \cdots x_m) = \sum_{i=1}^{n} c_i \prod_{j=1}^{m} x_j^{a_{ij}}$$

where $c_i > 0$ *and* a_{ij} *are real numbers.*

Recall the A-G Mean Inequality:

$$\boxed{\sum_i u_i \geq \prod_i \left(u_i/\delta_i\right)^{\delta_i}}$$

Letting $u_i = c_i \prod_j x_j^{a_{ij}}$, we obtain

$$g(x) = \sum_i c_i \prod_j x_j^{a_{ij}} \geq \prod_i \left[\frac{c_i \prod_j x_j^{a_{ij}}}{\delta_i}\right]^{\delta_i} = C(\delta) \prod_j x_j^{\alpha_{ij}(\delta)}$$

where $C(\delta) = \prod_i \left(c_i/\delta_i\right)^{\delta_i}$, $\alpha_j(\delta) = \sum_i a_{ij}\delta_i$

That is, we obtain a monomial approximation (lower bound) of the posynomial,

$$g(x) = \sum_i c_i \prod_j x_j^{a_{ij}} \ge C(\delta) \prod_j x_j^{\alpha_{ij}(\delta)}$$

where $$C(\delta) = \prod_i \left({}^{c_i}\!/\!{}_{\delta_i} \right)^{\delta_i}, \quad \alpha_j(\delta) = \sum_i a_{ij}\delta_i$$

which is exact when

$$\frac{c_1 \prod_j x_j^{a_{1j}}}{\delta_1} = \frac{c_2 \prod_j x_j^{a_{2j}}}{\delta_2} = \cdots = \frac{c_n \prod_j x_j^{a_{nj}}}{\delta_n}$$

Signomial Functions

coefficients not restricted in sign!

$$g(x_1, x_2, \cdots x_m) = \sum_{i=1}^{n} c_i \prod_{j=1}^{m} x_j^{a_{ij}}$$

Condensation has long been used in solving Signomial GP problems (which are essentially nonconvex) by means of a sequence of approximating Posynomial GP problems (which are essentially convex problems).

Example

Minimize x_1

subject to

$$(x_1 - 2)^2 + (x_2 - 4)^2 \geq 4$$

X is outside a circle centered at (2,4) with radius 2

$$(x_1 - 3)^2 + (x_2 - 3)^2 \leq 4$$

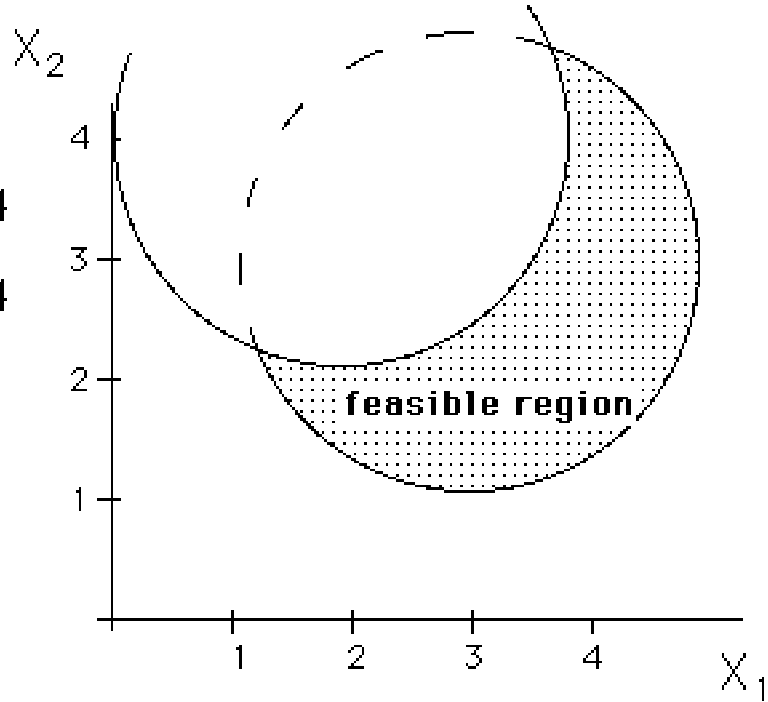
X is within a circle centered at (3,3) with radius 2

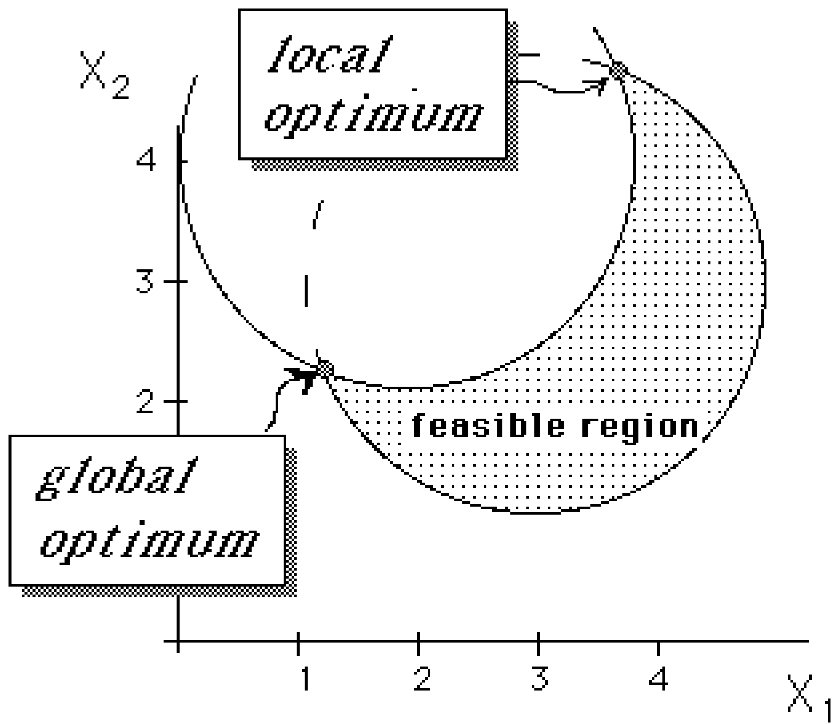
Minimize x_1

subject to

$$(x_1 - 2)^2 + (x_2 - 4)^2 \geq 4$$

$$(x_1 - 3)^2 + (x_2 - 3)^2 \leq 4$$




local optimum
global optimum
feasible region
X_2
X_1
1
2
3
4
2
3
4

Reformulation as a GP problem

constraint # 1

$$(X_1 - 2)^2 + (X_2 - 4)^2 \geq 4$$

$$\Rightarrow \quad (x_1^2 - 4x_1 + 4) + (x_2^2 - 8x_2 + 16) \geq 4$$

$$\Rightarrow \quad - x_1^2 + 4x_1 - x_2^2 + 8x_1 \leq 16$$

The constraint becomes the signomial constraint

$$\Rightarrow \quad \boxed{\frac{X_1}{4} + \frac{X_2}{2} - \frac{X_1^2}{16} - \frac{X_2^2}{16} \leq 1}$$

Reformulation as a GP problem

constraint # 2

$$(X_1 - 3)^2 + (X_2 - 3)^2 \leq 4$$

$$\Rightarrow \quad (x_1^2 - 6x_1 + 9) + (x_2^2 - 6x_2 + 9) \leq 4$$

$$\Rightarrow \quad x_1^2 - 6x_1 + x_2^2 + 14 \leq 6x_2$$

The constraint becomes the signomial constraint

$$\Rightarrow \quad \boxed{\frac{X_1^2 X_2^{-1}}{6} + \frac{X_2}{6} + \frac{7X_2^{-1}}{3} - X_1 X_2^{-1} \leq 1}$$

Signomial Geometric Program

Minimize X_1

subject to

$$\frac{X_1}{4} + \frac{X_2}{2} - \frac{X_1^2}{16} - \frac{X_2^2}{16} \le 1$$

$$\frac{X_1^2 X_2^{-1}}{6} + \frac{X_2}{6} + \frac{7X_2^{-1}}{3} - X_1 X_2^{-1} \le 1$$

$$X_1 > 0,\ X_2 > 0$$

To condense the signomial constraint

$$\frac{X_1}{4} + \frac{X_2}{2} - \frac{X_1^2}{16} - \frac{X_2^2}{16} \le 1$$

we first write it in the form

$$\frac{X_1}{4} + \frac{X_2}{2} \le 1 + \frac{X_1^2}{16} + \frac{X_2^2}{16}$$

$$\Rightarrow \frac{\dfrac{X_1}{4} + \dfrac{X_2}{2}}{1 + \dfrac{X_1^2}{16} + \dfrac{X_2^2}{16}} \le 1 \Rightarrow \frac{0.25X_1 + 0.5X_2}{1 + 0.0625\ X_1^2 + 0.0625\ X_2^2} \le 1$$

We next condense the denominator of

$$\frac{0.25X_1 + 0.5X_2}{1 + 0.0625\ X_1^2 + 0.0625\ X_2^2} \le 1$$

into a single term. Let's use the point $X_0 = (4,5)$ at which the terms of the denominator are

$$1 \ + \ 1 \ + \ 1.5626 \ = \ 3.5625$$

Then

$$\delta_1 = \delta_2 = \frac{1}{3.5625} = 0.2807 \quad \text{and} \quad \delta_3 = \frac{1.5625}{3.5625} = 0.4386$$

$$\delta_1 = \delta_2 = 0.2807, \quad \delta_3 = 0.4386$$

Coefficient:

$$\boxed{C(\delta) = \prod_{i=1}^{3}\left(\frac{c_i}{\delta_i}\right)^{\delta_i}}$$

$$C(\delta) = \left(\frac{1}{0.2807}\right)^{0.2807}\left(\frac{0.0625}{0.2807}\right)^{0.2807}\left(\frac{0.0625}{0.4386}\right)^{0.4386}$$

$$= 0.3987$$

$$\delta_1 = \delta_2 = 0.2807, \quad \delta_3 = 0.4386$$

Exponents:

$$\boxed{\mathbf{a}_j(\delta) = \sum_{i=1}^{3} \mathbf{a}_{ij}\delta_i}$$

$$\mathbf{a}_1 = 0\delta_1 + 2\delta_2 + 0\delta_3 = 2(0.2807) = 0.5614$$

$$\mathbf{a}_2 = 0\delta_1 + 0\delta_2 + 2\delta_3 = 2(0.4386) = 0.8772$$

$$\left.\begin{aligned} C(\delta) &= 0.3987 \\ a_1 &= \mathbf{0.5614} \\ a_2 &= \mathbf{0.8772} \end{aligned}\right\}$$

Condensed denominator is

$$\mathbf{0.3987}\, X_1^{0.5614}\, X_2^{0.8772}$$

monomial!!

Geometric Inequality implies

$$1 + 0.0625X_1^2 + 0.0625\ X_2^2 \quad \geq \quad 0.3987\ X_1^{0.5614}\ X_2^{0.8772}$$

and so

$$\frac{0.25X_1 + 0.5X_2}{1 + 0.0625\ X_1^2 + 0.0625\ X_2^2} \leq \frac{0.25X_1 + 0.5X_2}{0.3987\ X_1^{0.5614}\ X_2^{0.8772}}$$

$$\frac{posynomial}{monomial} = posynomial$$

$$\frac{0.25X_1 + 0.5X_2}{0.3987\ X_1^{0.5614}\ X_2^{0.8772}}$$

$$= \frac{0.25}{0.3987} X_1^{1-0.5614}\ X_2^{-0.8772} + \frac{0.5}{0.3987} X_1^{-0.5614}\ X_2^{1-0.8772}$$

$$= 0.627\ X_1^{0.4386}\ X_2^{-0.8772} + 1.254\ X_1^{-0.5614}\ X_2^{0.1228}$$

which is a posynomial!

If we constrain this posynomial so as to be ≤ 1, then by the geometric inequality, the original signomial should also be ≤ 1.

That is, any X feasible in the posynomial constraint derived by condensation will also be feasible in the signomial constraint:

$$\frac{0.25X_1 + 0.5X_2}{1 + 0.0625\,X_1^2 + 0.0625\,X_2^2}$$

$$\leq\ 0.627\,X_1^{0.4386}\,X_2^{-0.8772} + 1.254\,X_1^{-0.5614}\,X_2^{0.1228}\ \leq\ 1$$

The second signomial constraint may be condensed in a similar fashion:

$$\frac{X_1^2 X_2^{-1}}{6} + \frac{X_2}{6} + \frac{7X_2^{-1}}{3} - X_1 X_2^{-1} \leq 1$$

$$\Longrightarrow \quad \frac{X_1^2 X_2^{-1}}{6} + \frac{X_2}{6} + \frac{7X_2^{-1}}{3} \leq 1 + X_1 X_2^{-1}$$

$$\Longrightarrow \quad \frac{\dfrac{X_1^2 X_2^{-1}}{6} + \dfrac{X_2}{6} + \dfrac{7X_2^{-1}}{3}}{1 + X_1 X_2^{-1}} \leq 1$$

$$\frac{\dfrac{X_1^2X_2^{-1}}{6}+\dfrac{X_2}{6}+\dfrac{7X_2^{-1}}{3}}{1+X_1X_2^{-1}} \leq 1$$

At (4,5), the denominator is 1 + 0.8 = 1.8, so

$$\delta_1 = \frac{1}{1.8} = 0.555,\ \delta_2 = \frac{0.8}{1.8} = 0.444$$

can be condensed (using $\delta_1 = 0.555$, $\delta_2 = 0.444$) into the posynomial constraint

$$0.08385X_1^{1.555}X_2^{-0.555} + 0.08385X_1^{-0.444}X_2^{1.444} + 1.174X_1^{-0.444}X_2^{-0.555} \leq 1$$

The signomial GP problem is therefore approximated by the posynomial problem:

Minimize X_1

subject to

$$0.627\, X_1^{0.4386}\, X_2^{-0.8772} + 1.254\, X_1^{-0.5614}\, X_2^{0.1228} \le 1$$

$$0.08385 X_1^{1.555} X_2^{-0.555} + 0.08385 X_1^{-0.444} X_2^{1.444} + 1.174 X_1^{-0.444} X_2^{-0.555} \le 1$$

$$X_1 > 0,\; X_2 > 0$$

Monomial Method

We wish to find a *(positive)* solution of the following system of nonlinear (signomial) equations:

$$g_k(x) = \sum_i \sigma_{ik} c_{ik} \prod_j x_j^{a_{ijk}} = 0, \; k=1, \cdots N$$

where $\sigma_{ik} \in \{+1, -1\}$, $c_{ik} > 0$

Example:

$$\begin{cases} 2.5\, x_1^{-1.5} + 15\, x_1^{8/3} x_2^{-2} - 30 x_2 = 0 \\ 77 + 9\, x_2^{-1} - 28 x_1 x_2 - 4 x_1^{-3} = 0 \end{cases}$$

Define the index sets of the positive & negative terms of each equation:

$$T_k^+ = \{\ i \mid \sigma_{ik} > 0\} \ \& \ T_k^- = \{\ i \mid \sigma_{ik} < 0\}$$

Then separate each signomial into positive & negative parts:

$$g_k(x) = P_k(x) - Q_k(x)$$

where

$$P_k(x) = \sum_{i \in T_k^+} c_{ik} \prod_j x_j^{a_{ijk}} \ \& \ Q_k(x) = \sum_{i \in T_k^-} c_{ik} \prod_j x_j^{a_{ijk}}$$

$$g_k(x) = P_k(x) - Q_k(x) = 0$$

$$\Rightarrow \qquad P_k(x) = Q_k(x)$$

$$\Rightarrow \qquad \frac{P_k(x)}{Q_k(x)} = 1$$

Each of the posynomials $P_k(x)$ and $Q_k(x)$ are then condensed into monomial approximations $\overline{P}_k(x)$ and $\overline{Q}_k(x)$, respectively, and the ratio of the two monomials is also a monomial!

Each nonlinear equation is then approximated by a monomial equation

$$\boxed{\frac{P_k(x)}{Q_k(x)} \approx \frac{\overline{P}_k(x)}{\overline{Q}_k(x)} = C_k(\delta)\prod_j x_j^{\alpha_{jk}(\delta)} = 1}$$

for some choice of the weights (δ)

By taking the logarithms of both sides and making the change of variable $z_j = \ln x_j$

we get the linear equation

$$\boxed{\sum_j \alpha_{jk}(\delta)\, z_j = -C_k(\delta)}$$

0 Select an initial starting point x°.

1 Evaluate the weights of all the terms:

$$\delta_{ik} = \frac{c_{ik} \prod_j (x^{\circ})_j^{a_{ijk}}}{P_k(x^{\circ})} \ \forall\ i \in T_k^{+} \ \ \&\ \ \delta_{ik} = \frac{c_{ik} \prod_j (x^{\circ})_j^{a_{ijk}}}{Q_k(x^{\circ})} \ \forall\ i \in T_k^{-}$$

2 Evaluate $C_k(\delta)$ and $\alpha_{kj}(\delta)$

3 Solve the linear system of equations in z.

4 Exponentiate z to obtain x' (yielding x' >0!)

5 Test for convergence, e.g.,

$$\|x^{\circ} - x'\| \leq \varepsilon$$

If the test fails, replace x° with x' and return to step 1.

It can be shown that the "Monomial" Method is equivalent to Newton's Method applied to

$$\boxed{\ln\left[\frac{P_k(e^X)}{Q_k(e^X)}\right] = 0, \quad k = 1, \ldots N}$$

Standard Newton

$$P(x) - Q(x) = 0$$

Newton-Central

$$P(e^z) - Q(e^z) = 0$$

$$\frac{P(e^z)}{Q(e^z)} = 1$$

Monomial

$$\ln\left[\frac{P(e^z)}{Q(e^z)}\right] = 0$$

These all have the property that they will exactly follow the central path and yield strictly positive iterates!

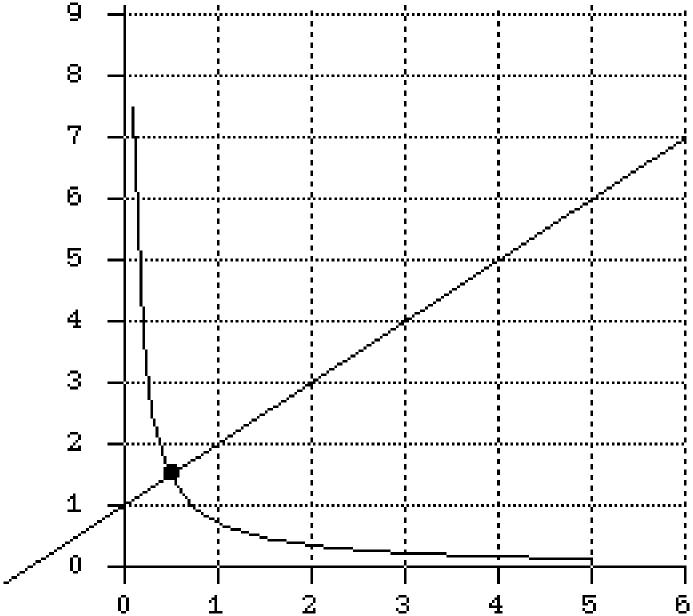
Example

A "toy" LCP:

$$y = Mx + q, \quad xy=0$$

$$\begin{cases} xy = \mu \\ y = x + 1 \end{cases}$$

i.e., one "complementarity" equation
one linear equation



$$\begin{cases} xy = \mu = 0.75 \\ y = x + 1 \end{cases}$$

In general, the solution is

$$x(\mu) = -\frac{1}{2} + \sqrt{\frac{1}{4} + \mu}$$

$$y(\mu) = \frac{1}{2} + \sqrt{\frac{1}{4} + \mu}$$

Monomial Method

Note that the "complementarity" equation is already in monomial form.

The linear equation is approximated by a monomial as follows:

$$P - Q = (x+1) - y = 0$$

$$\Rightarrow \frac{P}{Q} = \frac{x+1}{y} = 1$$

$$x + 1 \geq \left(\frac{x}{\delta_1}\right)^{\delta_1}\left(\frac{1}{\delta_2}\right)^{\delta_2} = \delta_1^{-\delta_1}\delta_2^{-\delta_2}x^{\delta_1}$$

where the weights are:

$$\delta_1 = \frac{x^{\circ}}{x^{\circ}+1}, \quad \delta_2 = \frac{1}{x^{\circ}+1}$$

The nonlinear system:

$$\begin{cases} xy = \mu \\ y = x + 1 \end{cases}$$

is approximated by the linear system:

$$\begin{cases} \ln x + \ln y = \ln \mu \\ \delta_1 \ln x - \ln y = \delta_1 \ln \delta_1 + \delta_1 \ln \delta_1 \end{cases}$$

that is,

$$\begin{cases} z_x + z_y = \ln \mu \\ \delta_1 z_x - z_y = \delta_1 \ln \delta_1 + \delta_1 \ln \delta_1 \end{cases}$$

where $z_x = \ln x, \quad z_y = \ln y$

In the **Monomial Method**, then, we solve

$$\begin{bmatrix} 1 & -1 \\ x^\circ/_{x^\circ+1} & -1 \end{bmatrix}\begin{bmatrix} z_x \\ z_y \end{bmatrix} = \begin{bmatrix} \ln \mu \\ C \end{bmatrix}$$

and update $x^\circ \leftarrow \exp(z_x)$ & $y^\circ \leftarrow \exp(z_y)$

while in **Newton's Method**, we solve

$$\begin{bmatrix} y^\circ & x^\circ \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \Delta_x \\ \Delta_y \end{bmatrix} = \begin{bmatrix} \mu - x^\circ y^\circ \\ y^\circ - x^\circ - 1 \end{bmatrix}$$

and update $x^\circ \leftarrow x^\circ + \Delta_x$ & $y^\circ \leftarrow y^\circ + \Delta_y$

Still another algorithm may be obtained by applying Newton's Method after making the logarithmic transformation:

$$\begin{cases} z_x + z_y = \ln \mu \\ e^{z_x} - e^{z_y} = -1 \end{cases}$$

which requires solving

$$\begin{bmatrix} 1 & 1 \\ \ln x^{\circ} & -\ln y^{\circ} \end{bmatrix} \begin{bmatrix} dz_x \\ dz_y \end{bmatrix} = \begin{bmatrix} \mu - x^{\circ} y^{\circ} \\ y^{\circ} - x^{\circ} - 1 \end{bmatrix}$$

and updating $z_x^{\circ} \leftarrow z_x^{\circ} + dz_x$ & $z_y^{\circ} \leftarrow z_y^{\circ} + dz_y$

Newton-Central

Newton's Method

$\mu = 10^{-8}$

stopping criterion $|\mu - xy| + |y - x - 1| \leq 10^{-8}$

Starting point: (100, 10)

k	x^k	y^k	$\mu - x^k y^k$	$y^k - x^k - 1$
0	1E2	1E1	¯1E3	¯9.1E1
1	8.18182E0	9.18182E0	¯7.5124E1	¯1.77636E¯15
2	3.85531E0	4.85531E0	¯1.87187E1	8.88178E¯16
3	1.70635E0	2.70635E0	¯4.618E0	4.44089E¯16
4	6.59832E¯1	1.65983E0	¯1.09521E0	0E0
5	1.8769E¯1	1.18769E0	¯2.22918E¯1	2.22045E¯16
6	2.5613E¯2	1.02561E0	¯2.6269E¯2	0E0
7	6.24066E¯4	1.00062E0	¯6.24445E¯4	0E0
8	3.9896E¯7	1E0	¯3.88961E¯7	0E0
9	1.00002E¯8	1E0	¯1.5129E¯13	0E0

Monomial Method

$$\mu = 10^{-8}$$

stopping criterion $|\mu - xy| + |y - x - 1| \leq 10^{-8}$

Starting point: (100, 10)

k	x^k	y^k	$\mu - x^k y^k$	$y^k - x^k - 1$
0	1E2	1E1	¯1E3	¯9.1E1
1	9.28919E¯5	1.07652E¯4	1.8198E¯23	¯9.99985E¯1
2	1.00076E¯8	9.99245E¯1	8.27181E¯24	¯7.55405E¯4
3	1E¯8	1E0	1.98523E¯23	¯2.88658E¯15

An Infeasible Path-Following Algorithm using the Newton-Central Method

Equations to be approximately solved at each iteration

$$\begin{cases} Ax - y = b \\ -Qx + A^T w + s = c \\ XSe = \mu e \\ WYe = \mu e \end{cases}$$

The logarithmic transformation is made, so that the complementarity equations are linearized, and the linear equations become nonlinear: $P(e^z) - Q(e^z) = 0$

An Infeasible Path-Following Algorithm using the Monomial Method

Equations to be approximately solved at each iteration

$$\begin{cases} Ax - y = \mathbf{b} \\ -Qx + A^T\mathbf{w} + \mathbf{s} = \mathbf{c} \\ XS\mathbf{e} = \mu\mathbf{e} \\ WY\mathbf{e} = \mu\mathbf{e} \end{cases}$$

The linear equations are approximated by monomial equations, and the logarithmic transformation is then made to linearize all the constraints.

0 Start with any interior solution $(x^\circ, y^\circ, s^\circ, w^\circ) > 0$ set $k=0$, and choose 3 tolerances $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$

1 Compute $\mu^k = \sigma \dfrac{x^k s^k + y^k w^k}{n+m}$, for $0 < \sigma < 1$

$$t_p^k = b + y^k - Ax^k, \quad \& \quad t_d^k = Qx^k + c - A^T w^k - s^k$$

2 If

$$\mu^k \le \varepsilon_1, \quad \frac{\|t_p^k\|}{\|b\| + 1} \le \varepsilon_2, \quad \& \quad \frac{\|t_d^k\|}{\|Qx^k + c\| + 1} \le \varepsilon_3$$

then stop & accept the current iterate as optimal.

3 Evaluate the weights

4 Compute coefficients & rhs of linear system

5 Solve linear system & return to step 1.

Properties of the sequence generated by this algorithm:

- *exactly on the central trajectory*
- *strictly positive*
- *converges if bounded and the algorithm does not fail*

Computational Experience

- Random subproblems with two variables, three constraints, and known solutions were randomly generated and used to build larger problems
- Separability was eliminated by performing a linear transformation.
- For each problem size, ten random test problems were tested.
- Initial solutions for Newton & Newton-Central algorithm are randomly generated but ON the central trajectory
- Initial solutions for Monomial algorithm are randomly generated but not on central trajectory

Number of Subproblems	Separable Problems		Nonseparable Problems	
	A	Q	A	Q
M=2	50%	25%	70%	100%
M=4	25%	12.5%	60%	100%
M=8	12.5%	6.25%	55%	100%
M=12	8.33%	4.17%	53.33%	100%
# variables	2M		3M	
# constraints	3M		5M	

Adjustment of factor σ^k

Standard Newton Algorithm:

$$\sigma^{k+1} = \begin{cases} \min\,(0.95,\ 1.3\sigma^k) & \text{if } \dfrac{\mu^{k+1}}{\mu^k} < 1 \\ \max\,(0.2,\ 0.7\sigma^k) & \text{otherwise} \end{cases}$$

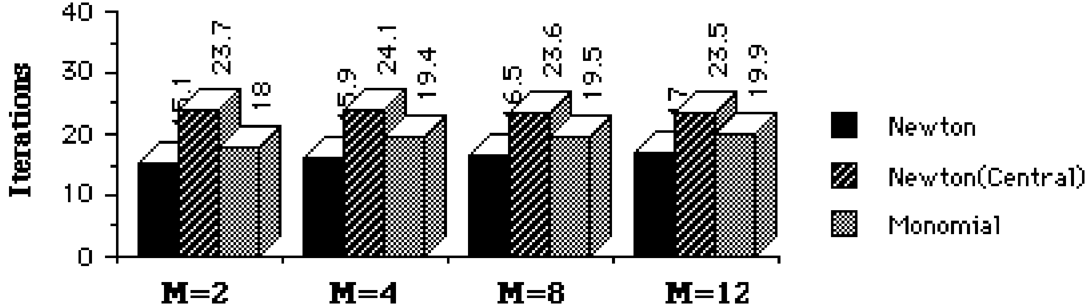
Newton-Central & Monomial Algorithms:

$$\sigma^{k+1} = \begin{cases} \min\,(0.95,\ 1.3\sigma^k) & \text{if } \dfrac{\mathrm{error}^{k+1}}{\mathrm{error}^k} < 1 \\ \max\,(0.2,\ 0.7\sigma^k) & \text{otherwise} \end{cases}$$

$$\mathrm{error}^k = \frac{t_p^k}{\|b\| + 1} + \frac{t_d^k}{\|Qx^k + c\| + 1}$$

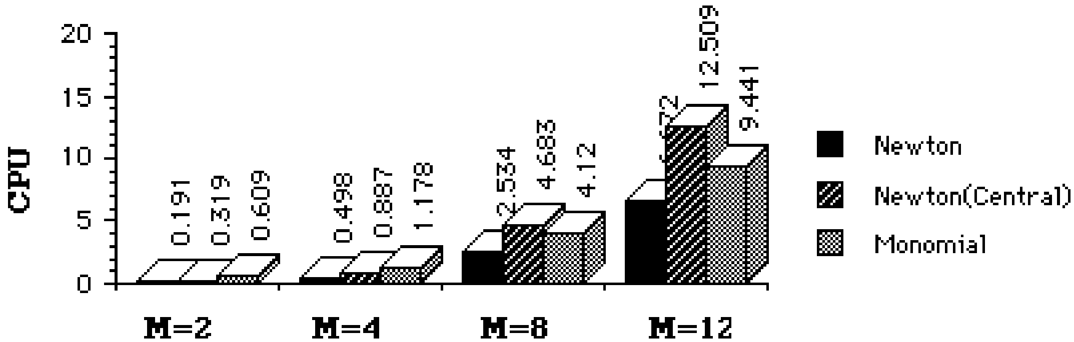
Separable Problems

Iterations vs # subproblems



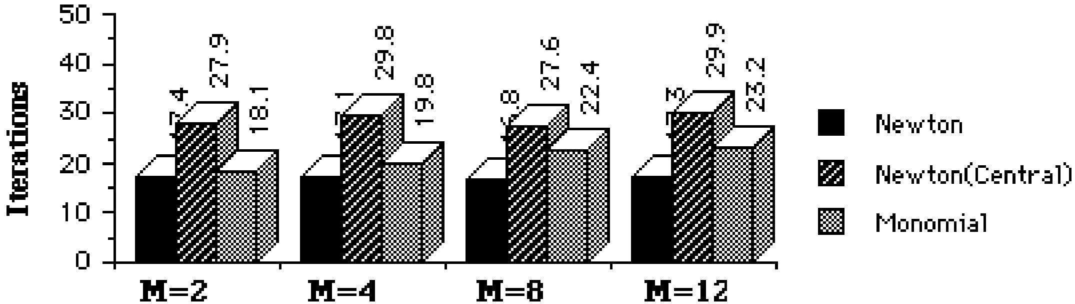
CPU vs # subproblems

Separable Problems



Nonseparable Problems

Iterations vs. # subproblems



Nonseparable Problems

CPU vs # subproblems