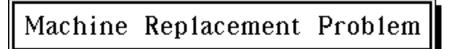


Machine Replacement Problem

At the beginning of each month, a machine is inspected and classified as:

- 1) Good as new
- 2) Operable, with minor deterioration
- 3) Operable, with major deterioration
- 4) Inoperable



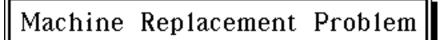
After determining the state of the machine, a decision must be made:

- 1) Keep the machine another month
- 2) Replace the machine with a new machine

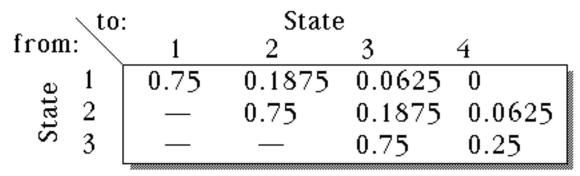
## Machine Replacement Problem

```
A replacement machine costs $3000,
minus trade-in value:
$1000 if in state 2
500 if in state 3
0 if in state 4
Monthly operating costs are $100, $200, and
$500 for a machine in states 1, 2, & 3,
```

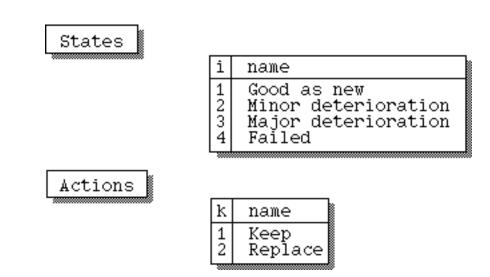
respectively.



### Survival probabilities



What is the optimal replacement policy?



@D. Bricker,U. of Iowa, 1998

#### page 7

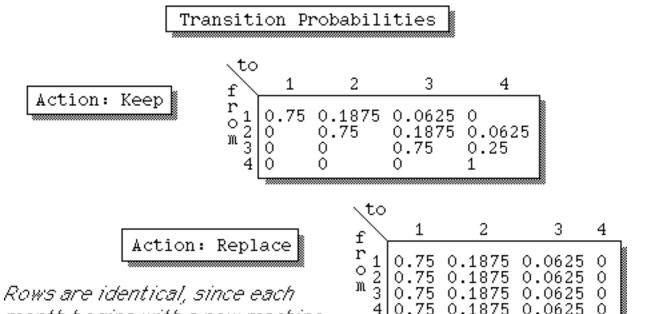
#### Cost Matrix

k	name	1	2	3	4
1	Keep	100	200	500	9999
2	Replace	9999	2100	2600	3100

(Rows ~ actions, Columns ~ states)

A value of 9999 above signals an infeasible action in a state.

### (includes cost of operating the new machine, if decision is to replace)



month begins with a new machine, regardless of current condition

# 🕼 Linear Programming Approach

Policy Iteration Method

# Linear Programming Model

What is the policy which minimizes the average cost/month in steady state?

Minimize  $\sum_{i=1}^{N} \mathbf{C}_{i}^{k} \mathbf{X}_{i}^{k}$ 

where

X<sup>k</sup><sub>i</sub> = probability that machine is in state #i and decision #k is selected <ு

					L	P Table	au
k:	1	1	2	1	2	2	R
i:	1	2	2	3	3	4	R H S
Min	100 0.25 -0.1875 -0.0625 1	0 0.25	-0.75 0.8125	500 0 0.25 1	-0.75 -0.1875	3100 -0.75 -0.1875 -0.0625 1	0 0 0 1

X<sup>k</sup><sub>i</sub> = probability that machine is in state #i and decision #k is selected

	ation	0	In	the mac	•	r: keep until ne fails
basic	<u>**</u>		*		*	,
k:	1 1	2	1	2	2	
_i:	12	2	3	3	4	rhs
Min	$\begin{array}{ccc} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{array}$		0 0 1 0	<sup>-304.878</sup> <sup>-1.17073</sup> <sup>-1.17073</sup> 2.73171 0.609756	0 0 0 1	-548.78 0.292683 0.292683 0.317073 0.097561
i∼st	ate,	k~action			ą o.	s is obtained by ne basic variable 'ate.

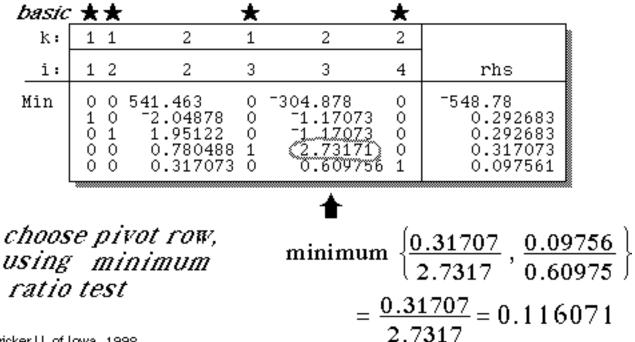
# Steadystate distribution resulting from this policy

#### Iteration 0

Policy: (Cost= 548.78 )

		1
State	Action	P{i}
1 Good as new 2 Minor deterioration 3 Major deterioration 4 Failed	1 Keep 1 Keep 1 Keep 2 Replace	0.292683 0.292683 0.317073 0.097561

### Choose a column having negative reduced cost, and enter it into the basis:



Iteration 1

	**			*	*	
k:	1 1	2	1	2	2	
i:	12	2	3	3	4	rhs
Min	$\begin{array}{ccc} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{array}$	628.571 -1.71429 2.28571 0.285714 0.142857	111.607 0.428571 0.428571 0.366071 -0.223214	0 0 0 1 0	0 0 0 1	<sup>-513.393</sup> 0.428571 0.428571 0.428571 0.116071 0.0267857

i~state, k~action

### $X_3^2$ has replaced $X_3^1$ in the basis

#### Iteration 1

Policy: (Cost= 513.393 )

State	Action	P{i}
1 Good as new	1 Keep	0.428571
2 Minor deterioration	1 Keep	0.428571
3 Major deterioration	2 Replace	0.116071
4 Failed	2 Replace	0.0267857

	**			*	*	
k:	1 1	2	1	2	2	
i:	12	2	3	3	4	rhs
Min	$\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{smallmatrix}$	628.571 -1.71429 2.28571 0.285714 0.142857	111.607 0.428571 0.428571 0.366071 -0.223214	0 0 0 1 0	0 0 0 1	-513.393 0.428571 0.428571 0.116071 0.0267857

i~state, k~action

Reduced costs are nonnegative... the optimality condition is satisfied/

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### Optimal Policy

State	Action
1 Good as new	1 Keep
2 Minor deterioration	1 Keep
3 Major deterioration	2 Replace
4 Failed	2 Replace

K⊅

Policy Iteration Method

🕼 Average cost per month

## Present value of all future costs



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### Machine Replacement Example

State	Action
1 Good as new	1 Keep
2 Minor deterioration	1 Keep
3 Major deterioration	1 Keep
4 Failed	2 Replace

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Policy Improvement Step: Evaluation of alternate actions

State #2, Minor deterioration

Current Policy: action #1, Keep

g(R)+Vi(R) = -992.683

k	name	C'	ΔC
1	Keep	-992.683	0
2	Replace	-451.22	541.463

*no improvement can be achieved by changing action in this state.* 

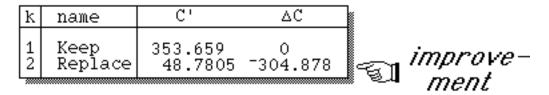
C'[k] = cost if action k is selected for one stage  $\Delta$ C[k] = improvement (if <0)

Policy Improvement Step: Evaluation of alternate actions

State #3, Major deterioration

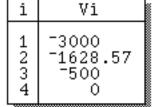
Current Policy: action #1, Keep

g(R) + Vi(R) = 353.659



C'[k] = cost if action k is selected for one stage $<math>\Delta C[k] = improvement (if < 0)$ 

State	Action		
1 Good as new 2 Minor deterioration 3 Major deterioration 4 Failed	1 Keep 1 Keep 2 Replace 2 Replace	T	new policy
Value Determinatio	20		



Policy Improvement Step: Evaluation of alternate actions

State #2, Minor deterioration

Current Policy: action #1, Keep

g(R) + Vi(R) = -1115.18

k	name	C'	ΔC
12	Keep	<sup>-</sup> 1115.18	0
	Replace	-486.607	628.571

*no improvement can be achieved by changing action in this state.* 

C'[k] = cost if action k is selected for one stage $<math>\Delta C[k] = improvement (if < 0)$ 

Policy Improvement Step: Evaluation of alternate actions

State #3, Major deterioration

Current Policy: action #2, Replace g(R)+Vi(R) = 13.3929

k	name	C'	ΔC
1	Keep	125	111.607
2	Replace	13.3929	0

*no improvement can be achieved by changing action in this state.* 

C'[k] = cost if action k is selected for one stage $<math>\Delta C[k] = improvement (if < 0)$ 

K⊅

The current policy is optimal!

### Minimizing: Present value of all future costs i.e., MDP with discounting is used.

# Assume a rate of return of 1.5% per month (18% per year)

$$\beta = \frac{1}{1+r} = \frac{1}{1.015} = 0.985222$$

That is, the present value of a \$1 cost next month is \$0.985222

### $\langle \mathcal{I} \rangle$

### Let's begin with an initial policy: keep machine until it fails i.e., R = (1, 1, 1, 2)

State		Action
1	Good as new	1 Keep
2	Minor deterioration	1 Keep
3	Major deterioration	1 Keep
4	Failed	2 Replace

Discount factor = 0.985222 (rate of return = 1.5%)

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## Value Determination

Solve the system of equations:  $\mathbf{v}_i(R) = \mathbf{C}_i^{k_i} + \beta \sum_{j \in S} \mathbf{p}_{ij}^{k_i} \, \mathbf{v}_j(R) \quad \forall \ i \in S$ 

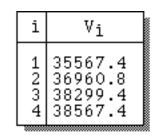
$$P^{R} = \begin{bmatrix} 0.75 & 0.1875 & 0.0625 & 0 \\ 0 & 0.75 & 0.1875 & 0.0625 \\ 0 & 0 & 0.75 & 0.25 \\ 0.75 & 0.1875 & 0.0625 & 0 \end{bmatrix}$$
$$\mathbf{v}_{i}(R) = \mathbf{C}_{i}^{k_{i}} + \beta \sum_{j \in S} p_{ij}^{k_{i}} \mathbf{v}_{j}(R) \quad \forall \ i \in S$$

$$\mathbf{v}_i(R) = \mathbf{C}_i^{k_i} + \beta \sum_{j \in S} \mathbf{p}_{ij}^{k_i} \mathbf{v}_j(R) \quad \forall i \in S$$

$$\begin{cases} \mathbf{v}_1 = \mathbf{100} + \mathbf{0.98522} \left( \mathbf{0.75v}_1 + \mathbf{0.1875v}_2 + \mathbf{0.0625v}_3 \right) \\ \mathbf{v}_2 = \mathbf{200} + \mathbf{0.98522} \left( \mathbf{0.75v}_2 + \mathbf{0.1875v}_3 + \mathbf{0.0625v}_4 \right) \\ \mathbf{v}_3 = \mathbf{500} + \mathbf{0.98522} \left( \mathbf{0.75v}_3 + \mathbf{0.25v}_4 \right) \\ \mathbf{v}_4 = \mathbf{3100} + \mathbf{0.98522} \left( \mathbf{0.75v}_1 + \mathbf{0.1875v}_2 + \mathbf{0.0625v}_3 \right) \end{cases}$$

Solution:

i	Vi	
1234	35567.4 36960.8 38299.4 38567.4	



That is, \$35,567.40 invested at 1.5% per month interest would sufficient to pay all future operation and replacement cost for the machine, if it is initially in state 1, i.e., "good as new"

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### Policy Improvement

Policy Improvement Step: Evaluation of alternate actions

State #2, Minor deterioration

Current Policy: action #1, Keep

Evaluate alternative action: #2, Replace

$$v'_{i} = C_{i}^{k'_{i}} + \beta \sum_{j} p_{ij}^{k'_{i}} v_{j}$$
 i=2, k'\_{i} =2

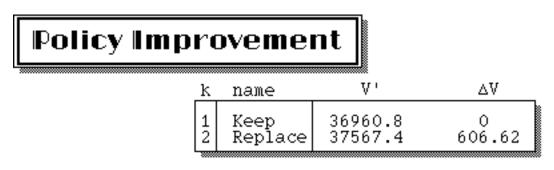
1 2 3

Vi		
35567.4 36960.8 38299.4 38567.4	$\mathbf{v}'_i = \mathbf{C}_i^{k'_i} + \beta \sum_j \mathbf{p}_{ij}^{k'_i} \mathbf{v}_j$	i=2, k <b>'</b> =2

$$\begin{split} \mathbf{v}_2' &= 2100 + 0.98522 \left( 0.75 \mathbf{v}_1 + 0.1875 \mathbf{v}_2 + 0.675 \mathbf{v}_3 \right) \\ &= 2100 + 0.98522 \left( 0.75 \!\times\!\! 35567.4 + 0.1875 \!\times\!\! 36960.8 \right. \\ &\quad + 0.0625 \!\times\!\! 38299.4 \Big) \end{split}$$

= 37567.40

That is, if we are initially in state 2 and replace the machine, but thereafter follow the original policy R, the present value of all future costs is \$27,567.40



V'(k) = total discounted cost if action k is selected for one stage, & current policy is followed thereafter

 $\Delta V(k) = improvement (if <0)$ 

Since  $v'_2 > v_2$ , the current policy for this state should not be changed.

# Policy Improvement

State #3, Major deterioration

Current Policy: action #1, Keep

Evaluate the alternate action: Replace

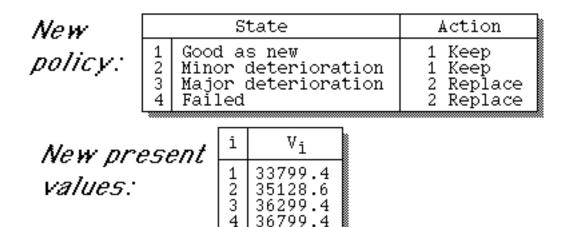
k	name	۷'	ΔV
1	Keep	38299.4	0
2	Replace	38067.4	-232.035

Since  $v'_3 < v_3$ , i.e.,  $\Delta v_3 = v'_3 - v_3 < 0$ , the policy in this state should be changed to "replace"

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### Value Determination

Discount factor = 0.985222 (rate of return = 1.5%)



# Policy Improvement

State #2, Minor deterioration

Current Policy: action #1, Keep

k	name	۷'	ΔV
1	Keep	35128.6	0
	Replace	35799.4	670.718

V'(k) = total discounted cost if action k is selected for one stage, & current policy is followed thereafter ∆V(k) = improvement (if <0)</pre>

The policy for this state should not be changed.

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## Policy Improvement

State #3, Major deterioration

Current Policy: action #2, Replace

k	name	Υ'	ΔV
1	Keep	36386.1	86.709
2	Replace	36299.4	0

V'(k) = total discounted cost if action k is selected for one stage, & current policy is followed thereafter ΔV(k) = improvement (if <0)</pre>

### The policy for this state should not be changed.



# No improvement is possible, so the current policy:

	State		Action	
1	Good as new	1	Keep	
2	Minor deterioration	1	Keep	
3	Major deterioration	2	Replace	
4	Failed	2	Replace	

### is optimal!

### K⊅