

Example:

Minimize $2x_1 + 5x_2 + 3x_3$

subject to

$$x_1 + 3x_2 + x_3 \geq 2$$

$$2x_1 - x_2 + x_3 \geq 2$$

$$x_j \in \{0,1\}, j=1,2,3$$

Lagrangian Relaxation of all inequality constraints:

$$L(x_1, x_2, x_3, I_1, I_2) = 2x_1 + 5x_2 + 3x_3 \\ + I_1 [2 - (x_1 + 3x_2 + x_3)] + I_2 [2 - (2x_1 - x_2 + x_3)]$$

That is,

$$L(x_1, x_2, x_3, I_1, I_2) = \\ (2 - I_1 - 2I_2)x_1 + (5 - 3I_1 + I_2)x_2 + (3 - I_1 - 2I_2)x_3 + 2I_1 + 2I_2$$

We will show that the corresponding Lagrangian dual is equivalent to the LP relaxation!

Lagrangian dual objective:

$$\hat{L}(I_1, I_2) = \text{Minimum}_{x_j \in \{0,1\}} L(x_1, x_2, x_3, I_1, I_2)$$

That is,

$$\begin{aligned} \hat{L}(I_1, I_2) = & 2I_1 + 2I_2 + \text{Minimum}_{x_j \in \{0,1\}} (2 - I_1 - 2I_2)x_1 \\ & + (5 - 3I_1 + I_2)x_2 + (3 - I_1 - 2I_2)x_3 \end{aligned}$$

The solution of the minimization is performed by assigning x_j the value 1 if its coefficient is positive, and 0 otherwise.

Thus, the dual objective function is

$$\hat{L}(\mathbf{I}_1, \mathbf{I}_2) = 2\mathbf{I}_1 + 2\mathbf{I}_2 + \min[0, 2 - \mathbf{I}_1 - 2\mathbf{I}_2] \\ + \min[0, 5 - 3\mathbf{I}_1 + \mathbf{I}_2] + \min[0, 3 - \mathbf{I}_1 - 2\mathbf{I}_2]$$

and the *Lagrangian dual* problem is to maximize this objective, i.e.,

$$\underset{\mathbf{I} \geq 0}{\text{Maximize}} \hat{L}(\mathbf{I}_1, \mathbf{I}_2)$$

The Lagrangian dual problem is equivalent to the following LP:

$$\text{Maximize } 2l_1 + 2l_2 + q_1 + q_2 + q_3$$

subject to

$$q_1 \leq 0 \ \& \ q_1 \leq 2 - l_1 - 2l_2$$

$$q_2 \leq 0 \ \& \ q_2 \leq 5 - 3l_1 + l_2$$

$$q_3 \leq 0 \ \& \ q_3 \leq 3 - l_1 - 2l_2$$

$$l_1 \geq 0, l_2 \geq 0$$

Make the change of variable $\phi_i = -\theta_i$:

$$\text{Maximize } 2I_1 + 2I_2 - j_1 - j_2 - j_3$$

subject to

$$I_1 + 2I_2 - j_1 \leq 2$$

$$3I_1 - I_2 - j_2 \leq 5$$

$$I_1 + 2I_2 - j_3 \leq 3$$

$$I_i \geq 0, i = 1, 2; j_j \geq 0, j = 1, 2, 3$$

The ordinary LP dual of this problem is:

$$\text{Minimize } 2x_1 + 5x_2 + 3x_3$$

subject to

$$x_1 + 3x_2 + x_3 \geq 2$$

$$2x_1 - x_2 + 2x_3 \geq 2$$

$$-x_1 \geq -1, \text{ i.e., } x_1 \leq 1$$

$$-x_2 \geq -1, \text{ i.e., } x_2 \leq 1$$

$$-x_3 \geq -1, \text{ i.e., } x_3 \leq 1$$

$$x_j \geq 0, \text{ } j=1,2,3$$

That is,

$$\text{Minimize } 2x_1 + 5x_2 + 3x_3$$

subject to

$$x_1 + 3x_2 + x_3 \geq 2$$

$$2x_1 - x_2 + 2x_3 \geq 2$$

$$0 \leq x_j \leq 1, \quad j=1,2,3$$

This is the *LP relaxation* of the original ILP!