Extending GP to Nonstandard Forms

This Hypercard stack was prepared by:
Dennis L. Bricker,
Dept. of Industrial Engineering,
University of Iowa,
Iowa City, Iowa 52242
e-mail: dbricker@icaen.uiowa.edu

author

In many cases, problems without signomial functions may be transformed into signomial GP problems:

Case 1

$$\underset{x>0}{\text{Minimize}} \quad y(x) = f(x) + [q(x)]^a\, h(x)$$

where the exponent a>0, and f(x), q(x), and h(x) are posynomials &/or signomials.

Make the substitution $z = q(x)$, and add the constraint $z \geq q(x) \Rightarrow z^{-1}\, q(x) \leq 1$

The problem becomes a GP problem:

$$\underset{x>0}{\text{Minimize}} \quad y(x) = f(x) + z^a\, h(x)$$

$$\text{subject to} \quad z^{-1}\, q(x) \leq 1$$

Example

$$\underset{x>0}{\text{Minimize}}\ (x_1^{-2} + x_2)^{0.8} + (x_1 + x_2^{-1})^{1.4}$$

Introduce two new variables x_3 and x_4 :

$$x_3 \geq x_1^{-2} + x_2 \quad \Rightarrow \quad x_1^{-2}\, x_3^{-1} + x_2\, x_3^{-1} \leq 1$$

$$x_4 \geq x_1 + x_2^{-1} \quad \Rightarrow \quad x_1\, x_4^{-1} + x_2^{-1}\, x_4^{-1} \leq 1$$

The problem becomes:

$$\underset{x>0}{\text{Minimize}}\ x_3^{0.8} + x_4^{1.4}$$

s.t.

$$x_1^{-2}\, x_3^{-1} + x_2\, x_3^{-1} \leq 1$$

$$x_1\, x_4^{-1} + x_2^{-1}\, x_4^{-1} \leq 1$$

which has zero degree of difficulty!

$$\underset{x>0}{\text{Minimize}}\ (x_1 - 2)^2 + (x_2 - 1)^2 + \frac{0.04}{1 - x_2^2 - \frac{x_1^2}{4}} + 5\,(1 - 2\,x_2 - x_1)^2$$

Expanding the squared expressions:

$$\underset{x>0}{\text{Minimize}}\ (x_1^2 - 4x_1 + 4) + (x_2^2 - 2x_2 + 1) \quad - \frac{0.04}{\frac{x_1^2}{4} + x_2^2 - 1}$$

$$+ 5\left(x_1^2 + 4x_2^2 + 4x_1\,x_2 - 2x_1 - 4x_2 + 1\right)$$

Example

Note the change of sign of the fraction!

Substitute a new variable x_3 for the denominator:

$$\underset{x>0}{\text{Minimize}}\; 6\,x_1^2 + 21x_2^2 + 20\,x_1\,x_2 - 14x_1 - 22\,x_2 + 10 - \frac{0.04}{x_3}$$

(The fraction should be as large as possible, and so the denominator should be as small as possible.)

To ensure equality, we choose to bound x_3 below:

$$x_3 \geq \frac{x_1^2}{4} + x_2^2 - 1 \Rightarrow \quad 0.25\,x_1^2\,x_3^{-1} + x_2^2\,x_3^{-1} - x_3^{-1} \leq 1$$

$$10 + \min_{x>0} \; 6\,x_1^2 + 21x_2^2 + 20\,x_1\,x_2 - 14x_1 - 22\,x_2 - 0.04x_3^{-1}$$

subject to

$$0.25\,x_1^2\,x_3^{-1} + x_2^2\,x_3^{-1} - x_3^{-1} \le 1$$

$$x_i > 0,\ i = 1, 2, 3$$

(degree of difficulty = 5)

Suppose instead that we had made substitutions for the quantities which were squared and for the denominator of the original fraction.

$$\underset{x>0}{\text{Minimize}}\ (x_1 - 2)^2 + (x_2 - 1)^2 + \frac{0.04}{1 - x_2^2 - \frac{x_1^2}{4}} + 5\,(1 - 2\,x_2 - x_1)^2$$

$$\underset{x>0}{\text{Minimize}}\ z_1^2 + z_2^2 + \frac{0.04}{z_3} + 5\,z_4^2$$

s.t.

$$z_1 \geq x_1 - 2 \Rightarrow x_1 z_1^{-1} - 2\,z_1^{-1} \leq 1$$

$$z_2 \geq x_2 - 1 \Rightarrow x_2 z_2^{-1} - 2\,z_2^{-1} \leq 1$$

$$z_3 \leq 1 - x_2^2 - \frac{x_1^2}{4} \Rightarrow z_3 + x_2^2 + 0.25\,x_1^2 \leq 1$$

$$z_4 \geq 1 - 2\,x_2 - x_1 \Rightarrow z_4^{-1} - 2\,x_2 z_4^{-1} - x_1\,z_4^{-1} \leq 1$$

$$x > 0,\ z > 0$$

Example

Note! The above re-formulation with

$$z_1 = x_1 - 2 \quad >0$$

$$z_2 = x_2 - 1 \quad >0$$

$$z_3 = 1 - x_2^2 - \frac{x_1^2}{4} \quad >0$$

$$z_4 = 1 - 2\, x_2 - x_1 \quad >0$$

assumes that the optimal x^* is such that

$$x_1^* > 2 \quad \& \quad 2\, x_2^* + x_1^* < 1$$

$$x_2^* > 1$$

which is infeasible!