Benders' Method

Farmer's Problem

A variation of Example problem in Birge & Louveaux, *Introduction to Stochastic Programming.*

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General Stochastic LP model:

$$Z = \min cx + \sum_{k=1}^{K} p_k q_k y_k$$

subject to

$$T_k x + W y_k = h_k, k = 1, \ldots K;$$

$$x \in X$$

(In the original problem, only matrix T was random, whereas in this version, q and h are also random.)

A farmer raises **wheat**, **corn**, and **sugar beets** on 500 acres of land. Before the planting season he wants to decide how much land to devote to each crop.

- wheat and corn are needed for **cattle feed**, which can be purchased from a wholesaler if not raised on the farm. *The amount needed depends upon the type of growing season.*
- Any grain in excess of the cattle feed requirement can be sold, with *the prices also dependent upon the type of growing season.*
- Up to 6000 tons of sugar beets can be sold for $36 per ton; any additional amounts can be sold for $10/ton.

In the original version of the problem, grain requirements for cattle and grain prices were not dependent upon the type of growing season.

First-stage data:

A,B=

1 1 1 < 500

i	variable	cost
1	Wheat acres	¯150
2	Corn acres	¯230
3	Beet acres	¯260

*The APL algorithm is designed to minimize, and so we will minimize the **negative** of the profits!*

Objective: Maximize

Second-stage data: K= # scenarios = 3

The following data vary by scenario: T, q, h

Costs:

i	variable	scenario: 1 (Good)	2 (Fair)	3 (Poor)
1	Wheat purchase	-220	-238	-245
2	Corn purchase	-195	-210	-215
3	Wheat sold	155	170	185
4	Corn sold	130	150	160
5	Subsidized beet sales	36	36	36
6	Unsubsidized beet sales	8	10	15

When yields are good, market prices are lowered, while cattle can supplement their diet with grass & fodder, requiring less grain.

```
Scenario # 1, with probability 0.33333333
Cost= ¯220 ¯195 155 130 36 8
T,W,h=
3 0      0 |  1 0 ¯1  0 0 0 >  190  Cattle rqmt for wheat
0 3.6    0 |  0 1  0 ¯1 0 0 >  230  Cattle rqmt for corn
0 0    ¯24 |  0 0  0  0 1 1 <    0
0 0      0 |  0 0  0  0 1 0 < 6000  Quota for beet subsidy
-----------------------------------------------------
Scenario # 2, with probability 0.33333333
Cost= ¯238 ¯210 170 150 36 10
T,W,h=
2.5 0    0 |  1 0 ¯1  0 0 0 >  200
0   3    0 |  0 1  0 ¯1 0 0 >  240
0   0  ¯20 |  0 0  0  0 1 1 <    0
0   0    0 |  0 0  0  0 1 0 < 6000
-----------------------------------------------------
Scenario # 3, with probability 0.33333333
Cost= ¯245 ¯215 185 160 36 15
T,W,h=
2 0      0 |  1 0 ¯1  0 0 0 >  210
0 2.4    0 |  0 1  0 ¯1 0 0 >  250
0 0    ¯16 |  0 0  0  0 1 1 <    0
0 0      0 |  0 0  0  0 1 0 < 6000
```

Iteration #1: Trial X for primal subproblems is

i	Variable	Value
1	Wheat acres	0
2	Corn acres	0
3	Beet acres	500

Primal subproblems summary: Second stage costs:

k	cost	p[k]
1	¯177350	0.33333333
2	¯158000	0.33333333
3	¯140800	0.33333333

First stage cost:	130000.00
Expected second stage cost:	¯158716.67
Total:	¯28716.67

Initial Upper Bound

Lagrangian multipliers

1	220	195	¯8	¯28
2	238	210	¯10	¯26
3	245	215	¯15	¯21
Sum	703	620	¯33	¯75

SUBPROBLEM DETAILS

Scenario #1 Optimal objective: ¯177350

i	variable	value
1	Wheat purchase	190
2	Corn purchase	230
5	Subsidized beet sales	6000
6	Unsubsidized beet sales	6000

Scenario #2 Optimal objective: ¯158000

i	variable	value
1	Wheat purchase	200
2	Corn purchase	240
5	Subsidized beet sales	6000
6	Unsubsidized beet sales	4000

Scenario #3 Optimal objective: ¯140800

i	variable	value
1	Wheat purchase	210
2	Corn purchase	250
5	Subsidized beet sales	6000
6	Unsubsidized beet sales	2000

The LP dual extreme points for each of the three scenarios are readily available from the LP solver:

Lagrangian multipliers

Scenario				
1	220	195	-8	-28
2	238	210	-10	-26
3	245	215	-15	-21
Sum	703	620	-33	-75

There are four dual variables, one for each of the constraints in the second-stage LP.

Estimating the Second-Stage Expected Costs:

For each scenario k, the dual information from the LP subproblem provides an *underestimating linear function* of $Q_k(x)$:

```
---------------------------------------
Scenario 1
 Cut             Lambda   Alpha
   1  ¯660 ¯702 ¯192 ¯81350
---------------------------------------
Scenario 2
 Cut             Lambda   Alpha
   1  ¯595 ¯630 ¯200 ¯58000
---------------------------------------
Scenario 3
 Cut             Lambda   Alpha
   1  ¯490 ¯516 ¯240 ¯20800
---------------------------------------
```

i.e.,
$-660X_1-702X_2-192X_3-81350$

Benders' *(Partial)* Master Problem

Minimize $150X_1 + 230X_2 + 260X_3 + \frac{1}{3}\theta_1 + \frac{1}{3}\theta_2 + \frac{1}{3}\theta_3$

subject to

$X_1 + X_2 + X_3 \le 500$ *(first-stage constraints)*

$$\begin{cases} \theta_1 \ge -660X_1 - 702X_2 - 192X_3 - 81350 \\ \theta_2 \ge -595X_1 - 630X_2 - 200X_3 - 58000 \\ \theta_3 \ge -490X_1 - 516X_2 - 240X_3 - 20800 \end{cases}$$

$X_j \ge 0,\ j = 1,2,3$

Solution of Master Problem

Initial Lower Bound

value= ¯269216.67

X= 500 0 0

First-stage cost: 75000

Estimated Q(X): ¯411350 ¯355500 ¯265800

Total (estimated) expected value: ¯269216.67

*This is an **underestimate** of the minimum expected cost*

(or, since the maximization problem was converted to a minimization problem, the maximum expected profit is bounded above by $269216.67.)

Iteration #2

Trial X for primal subproblems is

i	Variable	Value
1	Wheat acres	500
2	Corn acres	0
3	Beet acres	0

Primal subproblems summary: Second stage costs:

k	cost	p[k]
1	¯158200	0.33333333
2	¯128100	0.33333333
3	¯92400	0.33333333

First stage cost:	75000.00
Expected second stage cost:	¯126233.33
Total:	¯51233.33

	Lagrangian multipliers			
1	155	195	¯36	0
2	170	210	¯36	0
3	185	215	¯36	0
Sum	510	620	¯108	0

New Upper Bound

SUBPROBLEM DETAILS

Scenario #1 Optimal objective: ¯158200

i	variable	value
2	Corn purchase	230
3	Wheat sold	1310
10	slack_4	6000

Scenario #2 Optimal objective: ¯128100

i	variable	value
2	Corn purchase	240
3	Wheat sold	1050
10	slack_4	6000

Scenario #3 Optimal objective: ¯92400

i	variable	value
2	Corn purchase	250
3	Wheat sold	790
10	slack_4	6000

Scenario	Lagrangian multipliers
1	155 195 ¯36 0
2	170 210 ¯36 0
3	185 215 ¯36 0

Generating New Optimality Cuts

Scenario 1

Cut	Lambda			Alpha
1	¯660	¯702	¯192	¯81350
2	¯465	¯702	¯864	74300

i.e.,

$Q_k(x) \geq -660X_1 - 702X_2 - 192X_3 - 81350$

$Q_k(x) \geq -465X_1 - 702X_2 - 864X_3 - 74300$

Scenario 2

Cut	Lambda			Alpha
1	¯595	¯630	¯200	¯58000
2	¯425	¯630	¯720	84400

Scenario 3

Cut	Lambda			Alpha
1	¯490	¯516	¯240	¯20800
2	¯370	¯516	¯576	92600

Benders' *(Partial)* Master Problem

$$\text{Minimize } 150X_1 + 230X_2 + 260X_3 + \theta_1 + \theta_2 + \theta_3$$

subject to

$$X_1 + X_2 + X_3 \le 500 \quad \textit{(first-stage constraints)}$$

$$\begin{cases} \theta_1 \ge -660X_1 - 702X_2 - 192X_3 - 81350 \\ \theta_1 \ge -465X_1 - 702X_2 - 864X_3 + 74300 \\ \theta_2 \ge -595X_1 - 630X_2 - 200X_3 - 58000 \\ \theta_2 \ge -425X_1 - 630X_2 - 720X_3 + 84400 \\ \theta_3 \ge -490X_1 - 516X_2 - 240X_3 - 20800 \\ \theta_3 \ge -370X_1 - 516X_2 - 576X_3 + 92600 \end{cases}$$

$$X_j \ge 0, \;\; j = 1, 2, 3$$

Solution of Master Problem

New Lower Bound

```
value= ¯126373.36


X= 0 268.37798 231.62202
```

That is, plant 268.38 acres of corn and 231.62 acres of beets.

```
First-stage cost: 121948.66

Estimated Q(X): ¯314222.77 ¯251445.98 ¯179297.32

Total (estimated) expected value: ¯126373.36
```

This is a new underestimate of the minimum expected cost.

Therefore, $-126373.36 \leq Z^* \leq -51233.33$

i.e., the farmer can make a profit of at least $51233.33, but no more than $126373.36.

Iteration #3

Trial X for primal subproblems is

i	Variable	Value
1	Wheat acres	0.00000
2	Corn acres	268.37798
3	Beet acres	231.62202

Primal subproblems summary: Second stage costs:

k	cost	p[k]
1	¯254022.32	0.33333333
2	¯203937.95	0.33333333
3	¯145021.43	0.33333333

First stage cost:	121948.66
Expected second stage cost:	¯200993.90
Total:	¯79045.24

New Upper Bound

Lagrangian multipliers

1	220	130	¯36	0
2	238	150	¯36	0
3	245	160	¯36	0
Sum	703	440	¯108	0

SUBPROBLEM DETAILS

Scenario #1 Optimal objective: ¯254022.32

i	variable	value
1	Wheat purchase	190.00000
4	Corn sold	736.16071
5	Subsidized beet sales	5558.92857
10	slack_4	441.07143

Scenario #2 Optimal objective: ¯203937.95

i	variable	value
1	Wheat purchase	200.00000
4	Corn sold	565.13393
5	Subsidized beet sales	4632.44048
10	slack_4	1367.55952

Scenario #3 Optimal objective: ¯145021.43

i	variable	value
1	Wheat purchase	210.00000
4	Corn sold	394.10714
5	Subsidized beet sales	3705.95238
10	slack_4	2294.04762

Generating New Optimality Cuts

Scenario 1

Cut		Lambda		Alpha
1	¯660	¯702	¯192	¯81350
2	¯465	¯702	¯864	74300
3	¯660	¯468	¯864	71700

Scenario 2

Cut		Lambda		Alpha
1	¯595	¯630	¯200	¯58000
2	¯425	¯630	¯720	84400
3	¯595	¯450	¯720	83600

Scenario 3

Cut		Lambda		Alpha
1	¯490	¯516	¯240	¯20800
2	¯370	¯516	¯576	92600
3	¯490	¯384	¯576	91450

Solution of Master Problem

New Lower Bound

```
value= ¯114970.69

X= 117.12197 115.05964 267.81839
```

```
First-stage cost: 113664.79
Estimated Q(X): ¯290843.5 ¯230693.65 ¯164369.3
Total (estimated) expected value: ¯114970.69
```

Our bounds on the optimal solution are now:

$$-114970.69 \le Z^* \le -79045.24$$

Iteration #4

Trial X for primal subproblems is

i	Variable	Value
1	Wheat acres	117.12197
2	Corn acres	115.05964
3	Beet acres	267.81839

Primal subproblems summary: Second stage costs:

k	cost	p[k]
1	¯268380.76	0.33333333
2	¯224382.92	0.33333333
3	¯162931.42	0.33333333

First stage cost:	113664.79
Expected second stage cost:	¯218565.03
Total:	¯104900.24

New Upper Bound

Lagrangian multipliers

1	155	130	¯8	¯28
2	170	150	¯36	0
3	185	160	¯36	0
Sum	510	440	¯80	¯28

```
Primal subproblem details:
--------------------
Scenario #1 Optimal objective: ¯268380.76
 i   variable                          value
  3  Wheat sold                    161.36592
  4  Corn sold                     184.21471
  5  Subsidized beet sales        6000.00000
  6  Unsubsidized beet sales       427.64128
--------------------
Scenario #2 Optimal objective: ¯224382.92
 i   variable                           value
  3  Wheat sold                     92.804931
  4  Corn sold                     105.178922
  5  Subsidized beet sales        5356.367736
 10  slack_4                       643.632264
--------------------
Scenario #3 Optimal objective: ¯162931.42
 i   variable                           value
  3  Wheat sold                     24.243945
  4  Corn sold                      26.143138
  5  Subsidized beet sales        4285.094189
 10  slack_4                      1714.905811
```

SUBPROBLEM DETAILS

Solution of Master Problem

New Lower Bound

```
value= ¯106947.41

X= 175.61012 80 244.38988


First-stage cost: 108282.89
Estimated Q(X): ¯274671.56 ¯216595.01 ¯154424.32
Total (estimated) expected value: ¯106947.41
```

Our bounds on the optimal solution are now:

$$-106947.41 \leq Z^* \leq -104900.24$$

The gap between the upper & lower bounds on the profit has now been narrowed to $1047!

Iteration #5 Trial X for primal subproblems is

i	Variable	Value
1	Wheat acres	175.61012
2	Corn acres	80.00000
3	Beet acres	244.38988

Primal subproblems summary: Second stage costs:

k	cost	p[k]
1	¯270901.56	0.33333333
2	¯216595.01	0.33333333
3	¯154424.32	0.33333333

First stage cost:	108282.89
Expected second stage cost:	¯213973.63
Total:	¯105690.74

New Upper Bound

Lagrangian multipliers

1	155	130	¯36	0	
2	170	150	¯36	0	* *Repeated*
3	185	215	¯36	0	* *Repeated*
Sum	510	495	¯108	0	

Only one new "cut" is generated for the master problem!

Primal subproblem details:

Scenario #1 Optimal objective: ¯270901.56

i	variable	value
3	Wheat sold	336.83036
4	Corn sold	58.00000
5	Subsidized beet sales	5865.35714
10	slack_4	134.64286

Scenario #2 Optimal objective: ¯216595.01

i	variable	value
3	Wheat sold	239.0253
5	Subsidized beet sales	4887.7976
10	slack_4	1112.2024

Scenario #3 Optimal objective: ¯154424.32

i	variable	value
2	Corn purchase	58.00000
3	Wheat sold	141.22024
5	Subsidized beet sales	3910.23810
10	slack_4	2089.76190

SUBPROBLEM DETAILS

Solution of Master Problem

```
value= ¯106756.67

X= 170 80 250

First-stage cost: 108900
Estimated Q(X): ¯273140 ¯218250 ¯155580
Total (estimated) expected value: ¯106756.67
```

New Lower Bound

Our bounds on the optimal solution are now:

$$-106756.67 \leq Z^* \leq -105690.74$$

Iteration #6   Trial X for primal subproblems is

i	Variable	Value
1	Wheat acres	170
2	Corn acres	80
3	Beet acres	250

Primal subproblems summary: Second stage costs:

k	cost	p[k]
1	¯273140	0.33333333
2	¯218250	0.33333333
3	¯155580	0.33333333

First stage cost:	108900.00
Expected second stage cost:	¯215656.67
Total:	¯106756.67

Lagrangian multipliers

1	155	130	¯8	¯28	*Repeated!*
2	170	150	¯36	0	*Repeated!*
3	185	215	¯36	0	*Repeated!*
Sum	510	495	¯80	¯28	

New Upper Bound

Primal subproblem details:

Scenario #1 Optimal objective: ¯273140

i	variable	value
3	Wheat sold	320
4	Corn sold	58
5	Subsidized beet sales	6000

Scenario #2 Optimal objective: ¯218250

i	variable	value
3	Wheat sold	225
5	Subsidized beet sales	5000
10	slack_4	1000

Scenario #3 Optimal objective: ¯155580

i	variable	value
2	Corn purchase	58
3	Wheat sold	130
5	Subsidized beet sales	4000
10	slack_4	2000

SUBPROBLEM DETAILS

Solution of Master Problem

Unchanged!

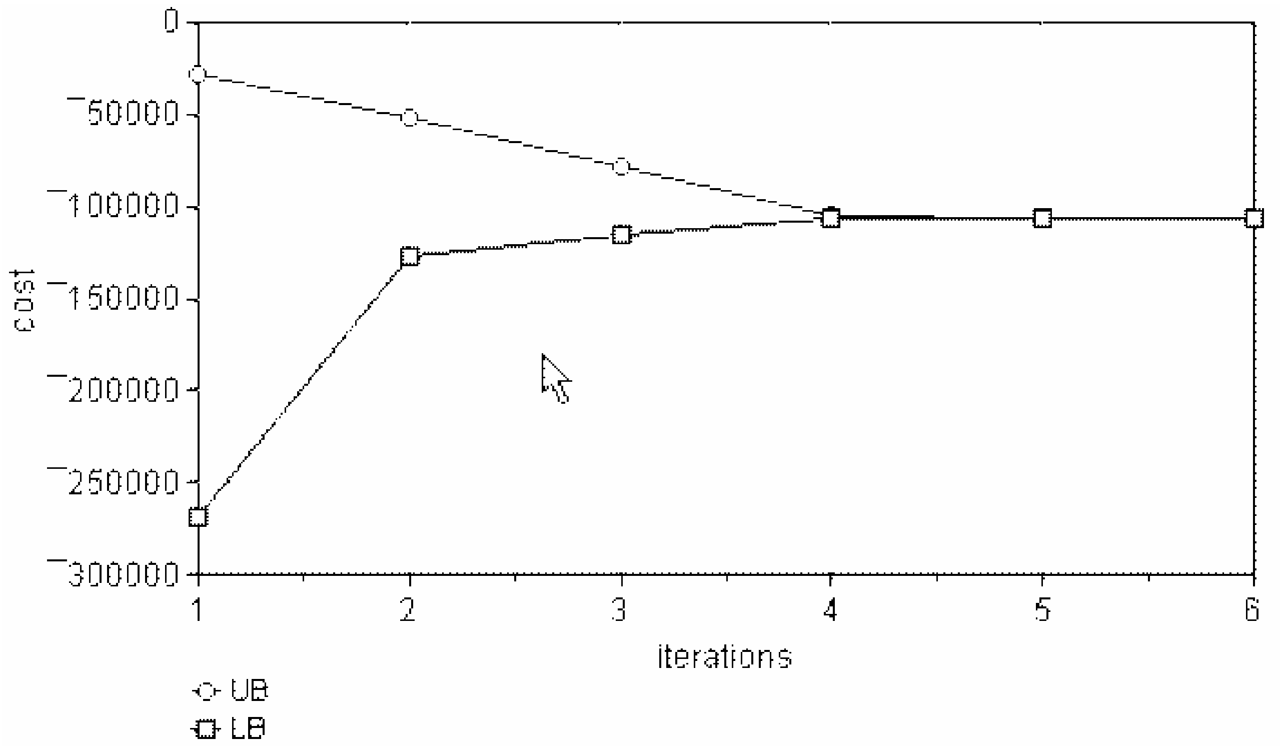
```
value= ¯106756.67

X= 170 80 250

First-stage cost: 108900
Estimated Q(X): ¯273140 ¯218250 ¯155580
Total (estimated) expected value: ¯106756.67


****************************************
```

Converged *at iteration #6!*

X was generated by previous master problem!

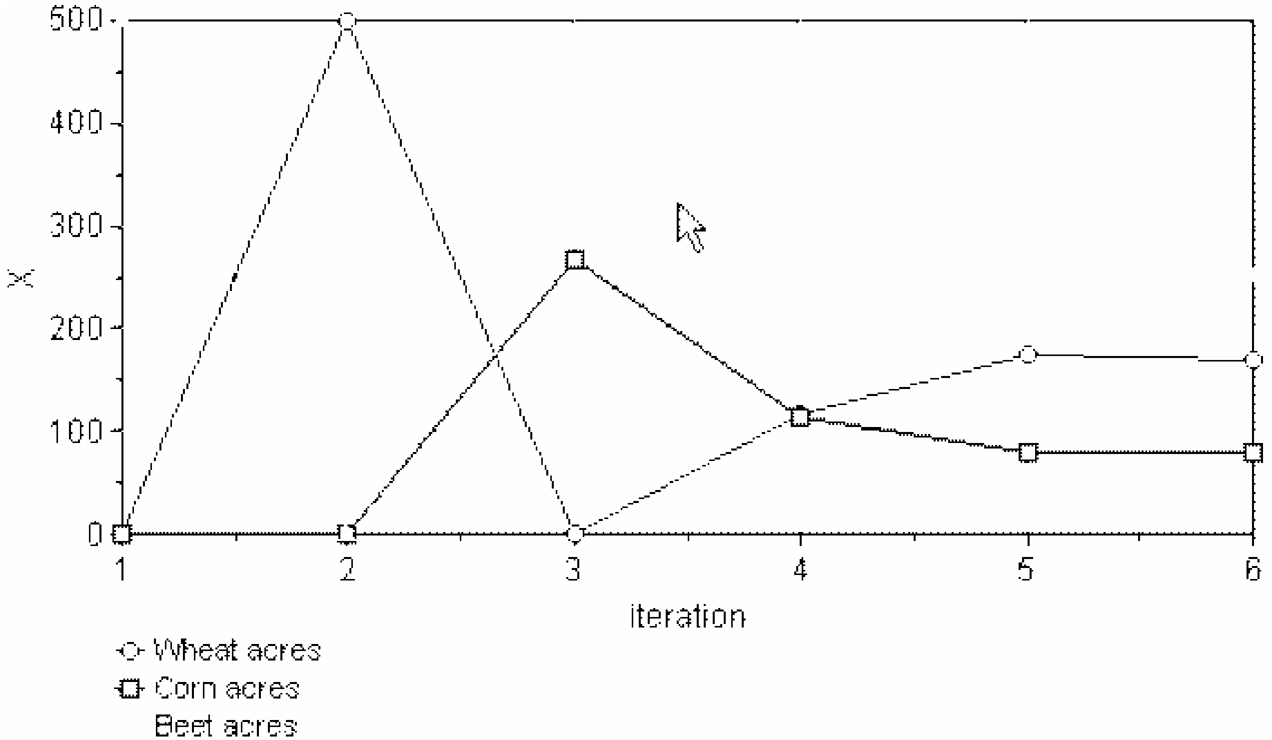
(The algorithm could have been terminated after the previous subproblem solution, since the gap between upper & lower bounds had then been reduced to zero!)

Plot of Convergence of Bounds



Convergence is monotonic, i.e., each upper bound is at least as good as the previous one, and likewise for lower bounds!

Plot of Convergence of First-Stage Decisions



Convergence of first-stage variables is not monotonic!

Final List of Optimality Cuts

Scenario 1

Cut	Lambda			Alpha
1	-660	-702	-192	-81350
2	-465	-702	-864	74300
3	-660	-468	-864	71700
4	-465	-468	-192	-108650
5	-465	-468	-864	59350
6	-465	-468	-192	-108650

Scenario 2

Cut	Lambda			Alpha
1	-595	-630	-200	-58000
2	-425	-630	-720	84400
3	-595	-450	-720	83600
4	-425	-450	-720	70000
5	-425	-450	-720	70000
6	-425	-450	-720	70000

Scenario 3

Cut	Lambda			Alpha
1	-490	-516	-240	-20800
2	-370	-516	-576	92600
3	-490	-384	-576	91450
4	-370	-384	-576	78850
5	-370	-516	-576	92600
6	-370	-516	-576	92600
