Expected Shortage & Surplus Functions

Cf. §3.3.3 "Expected shortage and surplus functions", in *Stochastic Programming*, by Willem K. Klein Haneveld and Maarten H. van der Vlerk, Dept of Econometrics & OR, University of Groningen, Netherlands

Definitions: Let ω be a *one-dimensional* random variable (e.g., *demand* for a commodity).

Then

- the **expected shortage function** is

$$H(x) \equiv E_\omega\left[(\omega - x)^-\right] \quad \text{for } x \in \mathbb{R}$$

- the **expected surplus function** is

$$G(x) \equiv E_\omega\left[(\omega - x)^+\right] \quad \text{for } x \in \mathbb{R}$$

where $z^+ \equiv \max\{z, 0\}$ and $z^- \equiv \max\{-z, 0\}$.

The **expected surplus function** is

$$G(x) \equiv E_D\left[(D-x)^+\right] \quad \text{for } x \in \mathbb{R}$$

In the case of a *discrete* distribution of *D*,

$$\begin{aligned} G(x) &= \sum_{d \geq x} (d-x)\, p_d \\ &= \sum_{d \geq x} d p_d - \sum_{d \geq x} x p_d \\ &= E\{D \mid d \geq x\} - x \times P\{D \geq x\} \end{aligned}$$

The **expected shortage function** is

$$H(x) \equiv E_D\left[(x-D)^+\right] \quad \text{for } x \in \mathbb{R}$$

In the case of a *discrete* distribution of *D*,

$$\begin{aligned} H(x) &= \sum_{d \le x} (x-d)\, p_d \\ &= \sum_{d \le x} x p_d - \sum_{d \le x} d p_d \\ &= x \times P\{D \le x\} - E\{D \mid d \le x\} \end{aligned}$$

Example:

Suppose that D has the ***discrete*** distribution:

Demand d	0	1	2	3
P_d	0.2	0.3	0.4	0.1

Expected surplus function

$$G(x) = E\{D \mid d \ge x\} - x \times P\{D \ge x\}$$

$$= \begin{cases} \left[(0\times0.2)+(1\times0.3)+(2\times0.4)+(3\times0.1)\right]-(0.2+0.3+0.4+0.1)x & \text{if } x \le 0 \\ \left[(1\times0.3)+(2\times0.4)+(3\times0.1)\right]-(0.3+0.4+0.1)x & \text{if } 0 \le x \le 1 \\ \left[(2\times0.4)+(3\times0.1)\right]-(0.4+0.1)x & \text{if } 1 \le x \le 2 \\ (3\times0.1)-(0.1)x & \text{if } 2 \le x \le 3 \\ 0 & \text{if } 3 \le x \end{cases}$$

$$= \begin{cases} \mu - x & \text{if } x \le 0 \\ 1.4-0.8x & \text{if } 0 \le x \le 1 \\ 1.1-0.5x & \text{if } 1 \le x \le 2 \\ 0.3-0.1x & \text{if } 2 \le x \le 3 \\ 0 & \text{if } 3 \le x \end{cases}$$

Expected shortage function

$$H(x) = x \times P\{D \le x\} - E\{D \mid d \le x\}$$

$$= \begin{cases} 0 \quad \text{if x} \le 0 \\ (0.2)x - [(0\times 0.2)] \qquad \text{if} \quad 0 \le \text{x} \le 1 \\ (0.2+0.3)x - [(0\times 0.2)+(1\times 0.3)] \qquad \text{if} \quad 1 \le \text{x} \le 2 \\ (0.2+0.3+0.4)x - [(0\times 0.2)+(1\times 0.3)+(2\times 0.4)] \qquad \text{if} \quad 2 \le \text{x} \le 3 \\ (0.2+0.3+0.4+0.1)x - [(0\times 0.2)+(1\times 0.3)+(2\times 0.4)+(3\times 0.1)] \qquad \text{if} \quad 3 \le \text{x} \end{cases}$$

$$= \begin{cases} 0 \quad \text{if x} \le 0 \\ 0.2x \qquad \text{if} \quad 0 \le \text{x} \le 1 \\ 0.5x - 0.3 \qquad \text{if} \quad 1 \le \text{x} \le 2 \\ 0.9x - 1.1 \qquad \text{if} \quad 2 \le \text{x} \le 3 \\ x - \mu \qquad \text{if} \quad 3 \le \text{x} \end{cases}$$

Properties:

$$H(x) = \int_{-\infty}^{x} F(t)\, dt$$

$$G(x) = \int_{x}^{\infty} \left[1 - F(t)\right] dt$$

$$H(x) - G(x) = x - E_\omega[\omega]$$

Define $\mu^- \equiv E_\omega\left[(\omega)^-\right]$ and $\mu^+ \equiv E_\omega\left[(\omega)^+\right]$.

Properties of Expected Shortage Function

H is *nonnegative*, *nondecreasing*, and *convex*

Under the assumption that $\mu^{-} < +\infty$,

- $H(x) < +\infty$ for all $x \in \mathbb{R}$
- H is continuous
- H is Lipschitz continuous with constant 1
- The left and right derivative of H exist everywhere and are given by

  left derivative: $H'_{-} = P\{\omega < x\}$

  and

  right derivative: $H'_{+}(x) = P\{\omega \le x\}$

- H is subdifferentiable, with subdifferential set (the interval)

$$\partial H(x) = \left[P\{\omega < x\}, P\{\omega \le x\} \right]$$

- H is differentiable at any continuity point x_0 of F with derivative

$$H'(x_0) = F(x_0)$$

- The curve $y = H(x)$ has a horizontal asymptote 0 at $-\infty$.

If $\mu^{+} \equiv E\left[(\omega)^{+}\right] < +\infty$, then

- The curve $y = H(x)$ has $\mathbf{x} - \mu$ as asymptote at $+\infty$,
- Outside the convex hull of the support of ω the curve $y = H(x)$ coincides with its asymptotes.

Properties of Expected Surplus Function

G is *nonnegative*, *nonincreasing*, and *convex*

Under the assumption that $\mu^+ < +\infty$,

- $G(x) < +\infty$ for all $x \in \mathbb{R}$
- G is continuous
- G is Lipschitz continuous with constant 1
- The left and right derivative of H exist everywhere and are given by

$$\textit{left derivative:}\quad G'_- = -P\{\omega \geq x\}$$

and

$$\textit{right derivative:}\, G'_+(x) = -P\{\omega > x\}$$

- G is subdifferentiable, with subdifferential set (the interval)

$$\partial G(x) = \left[-P\{\omega \geq x\}, -P\{\omega > x\}\right]$$

- G is differentiable at any continuity point x_0 of F with derivative

$$G'(x_0) = F(x_0) - 1$$

- The curve $y = G(x)$ has a horizontal asymptote 0 at $+\infty$.

If $\mu^{-} \equiv E_{\omega}\left[(\omega)^{-}\right] < +\infty$, then

- The curve $y = G(x)$ has $\mu - x$ as asymptote at $-\infty$,
- Outside the convex hull of the support of ω the curve $y = G(x)$ coincides with its asymptotes.

Define the one-dimensional **expected optimal value** function

$$Q(x) = E_\omega\left[\inf_y \ \{q_1 y_1 + q_2 y_2 : y_1 - y_2 = \omega - x; y_1 \geq 0, y_2 \geq 0\}\right]$$

Here q_1 is the cost per unit **surplus**, and

q_2 is the cost per unit **shortage**.

If $q_1 + q_2 > 0$, then the unique optimal solution to the LP problem defining Q is

$$\hat{y}_1 = (\omega - x)^+, \hat{y}_2 = (\omega - x)^-$$

so that

$$Q(x) = q_1 G(x) + q_2 H(x)$$

Summary: Assuming $q^+ + q^- > 0$, $\mu^+ < +\infty$, and $\mu^- < +\infty$,

Property	***H(x)***	***G(x)***	***Q(x)***
Monotonicity	nondecreasing	nonincreasing	
Left derivative	$P\{\omega < x\}$	$-P\{\omega \geq x\}$	$-q^+ + (q^+ + q^-) P\{\omega < x\}$
Right derivative	$P\{\omega \leq x\}$	$-P\{\omega > x\}$	$-q^+ + (q^+ + q^-) P\{\omega \leq x\}$
Derivative	$F(x)$	$F(x) - 1$	$-q^+ + (q^+ + q^-) F(x)$
Left asymptote	0	$\mu - x$	$q^+(\mu - x)$
Right asymptote	$x - \mu$	0	$q^-(x - \mu)$