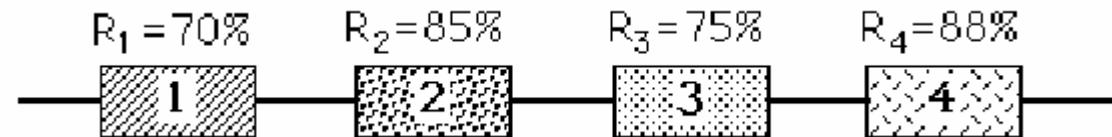


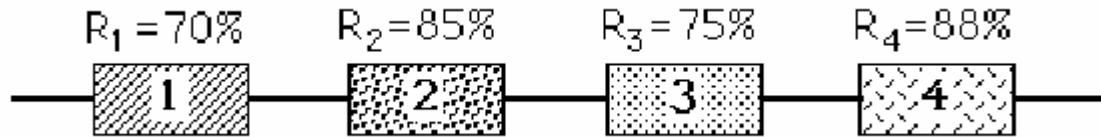
# Optimal Redundancy to Maximize System Reliability

- *Dynamic Programming Model*
- *Integer Programming Model*

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One of the systems of a communication satellite consists of four unreliable components each of which are necessary for successful operation of the satellite—the probabilities that a component survives the planned lifetime of the satellite (i.e., the *reliabilities*) are shown below:





Assuming that component failures are independent,

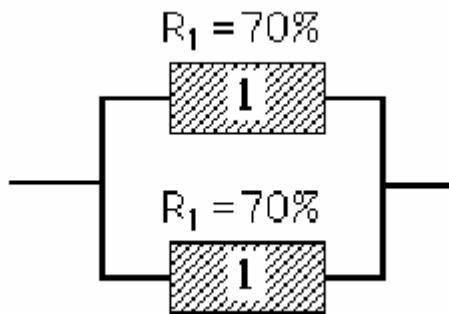
### Reliability of system

$$= P\{\text{components 1 through 4 survive}\}$$

$$= P\{\#1 \text{ survives}\} \times P\{\#2 \text{ survives}\} \times P\{\#3 \text{ survives}\} \times P\{\#4 \text{ survives}\}$$

$$= 0.70 \times 0.85 \times 0.75 \times 0.88 = \mathbf{39.27\%}$$

This is an unacceptably low system reliability, and so redundant units of one or more components will be used in the design.



The reliability of a component may be increased by including redundant units!

Reliability of component #1

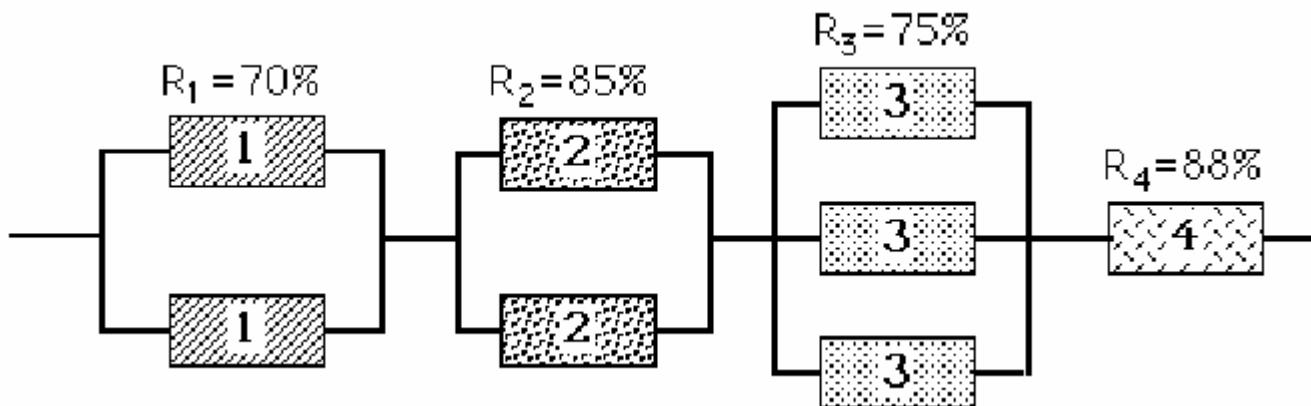
$$= P\{\text{at least one unit survives}\}$$

$$= 1 - P\{\text{both units fail}\}$$

$$= 1 - 0.30 \times 0.30 = 91\%$$

*This assumes what is referred to as “hot standby”, i.e., a standby unit may fail even before it is put into service!*

By using redundant units of each component, the system reliability can be dramatically increased—for example:



$$\begin{cases} \text{System} \\ \text{Reliability} \end{cases} = \left[ 1 - (0.30)^2 \right] \times \left[ 1 - (0.15)^2 \right] \times \left[ 1 - (0.25)^2 \right] \times [0.88] \\
 = 0.91 \times 0.9775 \times 0.984375 \times 0.88 = 77.0551\%$$

The problem faced by the designer is to maximize the system reliability, subject to a restriction on the total weight of the system.

Component	1	2	3	4
Weight (kg)	1	2	1	3

Total weight must not exceed 12 kg.

*(Total weight of one unit of each component is 7 kg, leaving 5 kg for redundant units.)*

## Reliability (%) vs. # redundant units

Component	1 unit	2 units	3 units
1	70	91	97.3
2	85	97.75	99.6625
3	75	93.75	98.4375
4	88	98.56	99.8272

*We will assume that no more than three units of any component will be included!*

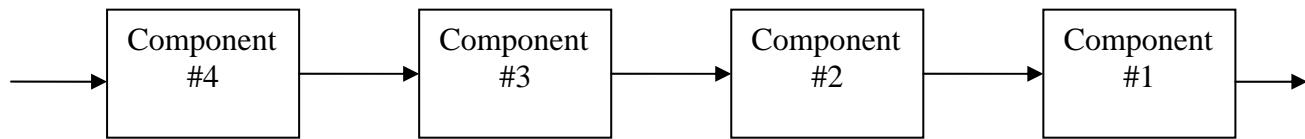
## Dynamic Programming Model

Stage:  $n$  component type

Decision:  $x_n$  # of units of component  $n$  included in system

State:  $s_n$  slack weight, i.e., # kg available

*We impose a sequential decision-making structure on the problem by supposing that we consider the components one at a time, deciding how many units to include based upon the available weight capacity.*

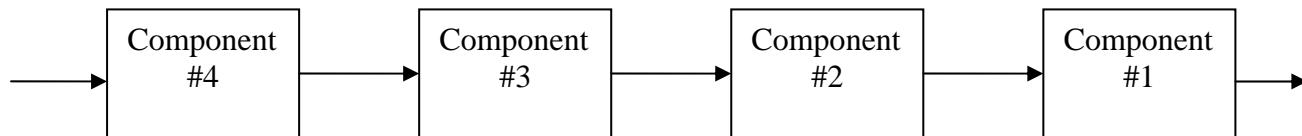


Arbitrarily we will use a “backward” order in what follows!

*That is, imagine that we first consider how many units of component #4 are to be included when we begin with 12 kg of available capacity, while component #1 is the last to be considered.*

## Optimal Value Function

$f_n(s_n)$  = maximum reliability of the subsystem consisting of devices n, n-1, ... 1, if  $s_n$  kg of available capacity remains to be allocated.



### Recursive definition of function

$$f_n(s_n) = \underset{1 \leq x_n \leq \frac{s_n}{w_n}}{\text{maximum}} \left\{ \left( 1 - p_n^{x_n} \right) \times f_{n-1}(s_n - w_n x_n) \right\}$$

$$f_0(s_0) = \begin{cases} 1 & \text{if } s_0 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

## APL function definition

```
v z←F N;t
[1] A
[2] A Optimal redundancy to maximize reliability
[3] A
[4] :if N=0
[5]   z←((ρs)ρ1),-BIG
[6] :else
[7] A Recursive definition of optimal value function
[8]   z←Maximize (((ρs)ρ1)∘.-{(1-R[N])×x})×(F N-1)[TRANSITION s∘.-W[N]×x
[9] :endif
    v
```

Component #1: reliability = 70%, weight = 1 kg.

<b>Stage 1</b>				
s \ x:	1	2	3	Maximum
1	0.7000	0.99999	0.99999	0.7000
2	0.7000	0.9100	0.99999	0.9100
3	0.7000	0.9100	0.9730	0.9730

*etc.*

Component #2: reliability = 80%, weight = 2 kg.

Stage 2

$s \setminus x:$	1	2	3	Maximum
3	0.5600	99.9999	99.9999	0.5600
4	0.7280	99.9999	99.9999	0.7280
5	0.7784	0.6720	99.9999	0.7784
6	0.7784	0.8736	99.9999	0.8736
7	0.7784	0.9341	0.6944	0.9341
8	0.7784	0.9341	0.9027	0.9341

etc.

For example, suppose that we have 6 kg of capacity remaining, i.e.,  $s_2 = 6$ , and we choose to include 2 units of component #2. Then we obtain 97.75% reliability of subsystem #2 and arrive at stage 1 (component #1) with  $6-2\times2=2$  kg of capacity remaining, so that we can achieve 91% reliability (  $f_1(2)=0.91$  ) in subsystem #1.

Hence the subsystem of components 1&2 will have reliability  $0.9775\times0.91 = 0.8736$

Component #3: reliability = 75%, weight = 1 kg.

Stage 3

s \ x:	1	2	3	Maximum
4	0.4200	0.9999	0.9999	0.4200
5	0.5460	0.5250	0.9999	0.5460
6	0.5838	0.6825	0.5513	0.6825
7	0.6552	0.7298	0.7166	0.7298
8	0.7006	0.8190	0.7662	0.8190
9	0.7006	0.8757	0.8600	0.8757

etc.

Component #4: reliability = 88%, weight = 3 kg.

Stage 4

s \ x:	1	2	3	Maximum
7	0.3696	0.9999	0.9999	0.3696
8	0.4805	0.9999	0.9999	0.4805
9	0.6006	0.9999	0.9999	0.6006
10	0.6422	0.4140	0.9999	0.6422
11	0.7207	0.5381	0.9999	0.7207
12	0.7706	0.6727	0.9999	0.7706

*Only the last row of this table need be computed to find the optimal reliability with 12 kg of capacity!*

## Summary of computations

### Stage 4

Current State	Optimal Decision	Optimal Value	Next State
cap 7	1 units	0.3696	cap 4
cap 8	1 units	0.4805	cap 5
cap 9	1 units	0.6006	cap 6
cap 10	1 units	0.6422	cap 7
cap 11	1 units	0.7207	cap 8
cap 12	1 units	0.7706	cap 9

### Stage 2

Current State	Optimal Decision	Optimal Value	Next State
cap 3	1 units	0.5600	cap 1
cap 4	1 units	0.7280	cap 2
cap 5	1 units	0.7784	cap 3
cap 6	2 units	0.8736	cap 2
cap 7	2 units	0.9341	cap 3
cap 8	2 units	0.9341	cap 4

### Stage 3

Current State	Optimal Decision	Optimal Value	Next State
cap 4	1 units	0.4200	cap 3
cap 5	1 units	0.5460	cap 4
cap 6	2 units	0.6825	cap 4
cap 7	2 units	0.7298	cap 5
cap 8	2 units	0.8190	cap 6
cap 9	2 units	0.8757	cap 7

### Stage 1

Current State	Optimal Decision	Optimal Value	Next State
cap 1	1 units	0.7000	cap 0
cap 2	2 units	0.9100	cap 0
cap 3	3 units	0.9730	cap 0
cap 4	3 units	0.9730	cap 1
cap 5	3 units	0.9730	cap 2
cap 6	3 units	0.9730	cap 3

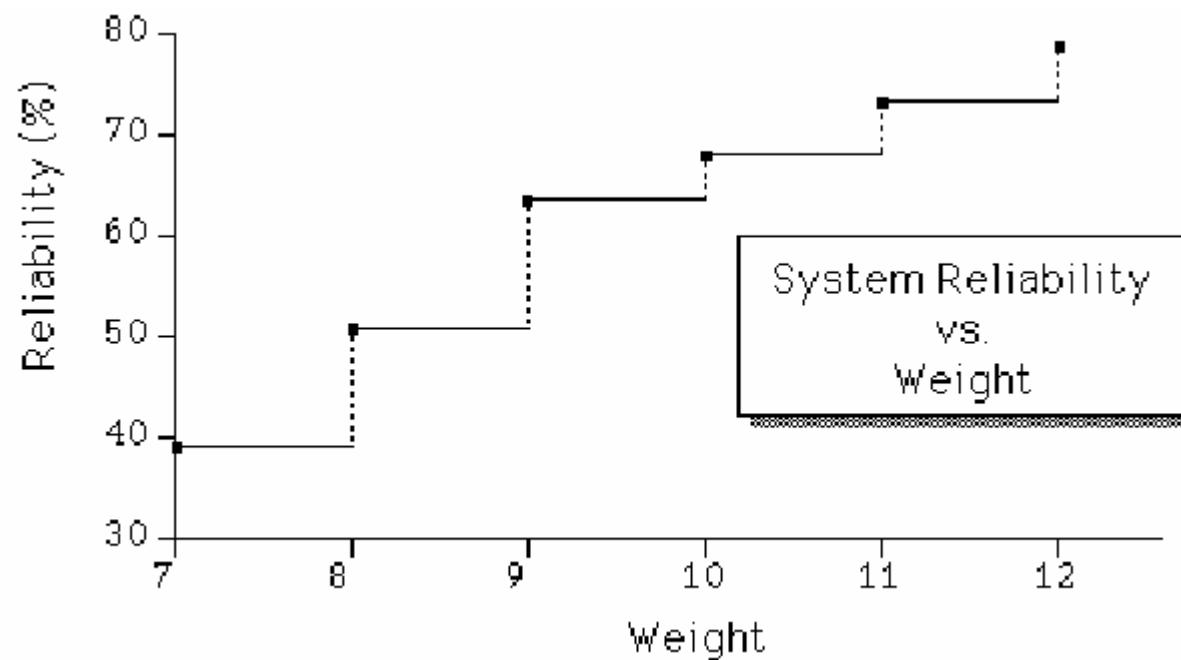
The maximum reliability, then, given a 12 kg weight restriction, is  $f_4(12) = 77.06\%$

By a “forward pass” through the tables, we can determine the optimal design:

<u>stage</u>	<u>state</u>	<u>decision</u>
4	cap 12	1 units
3	cap 9	2 units
2	cap 7	2 units
1	cap 3	3 units
0	cap 0	

*That is, the optimal design includes 1 of component #4, 2 each of components #2 & #3, and 3 of component #1.*

- What reduction in reliability would occur if the weight restriction were 11 kg rather than 12?
- What is the optimal design with a weight restriction of 11 kg?



## Integer Programming Model

Define *binary* decision variables:

$X_{in}$  = 1 if  $n$  units of component  $i$  are included  
in the system

$X_{in}$  = 0 otherwise

*Notation:*

Component i	$R_{i1}$	$R_{i2}$	$R_{i3}$
1	0.70	0.91	0.973
2	0.80	0.9775	0.996625
3	0.75	0.9375	0.984375
4	0.88	0.9856	0.998272

**Objective:**

In order to linearize the objective, we will instead maximize the ***logarithm of the reliability***:

$$Maximize \quad \sum_{i=1}^4 \sum_{n=1}^3 (\ln R_{in}) X_{in}$$

*subject to*

$$\sum_{i=1}^4 \sum_{n=1}^3 (W_i n) X_{in} \leq W_{\max}$$

$$\sum_{n=1}^3 X_{in} = 1 \quad \forall i = 1, 2, 3, 4$$

$$X_{in} \in \{0, 1\} \quad \forall i \& n$$

Component i	$\ln R_{i1}$	$\ln R_{i2}$	$\ln R_{i3}$
1	-0.35667	-0.094311	-0.02737
2	-0.22314	-0.040822	-0.008032
3	-0.28768	-0.064539	-0.01575
4	-0.12783	-0.014505	-0.001729

## LINGO model:

```
SETS:
  COMPONENT / A B C D/:
    WEIGHT;
  UNITS / 1..3/;
  LOG(COMPONENT,UNITS): LNR, X;
ENDSETS

DATA:
  WEIGHT = 1 2 1 3;
  WMAX = 12;
  LNR = -0.35667 -0.094311 -0.027371
        -0.22314 -0.040822 -0.0080322
        -0.28768 -0.064539 -0.015748
        -0.12783 -0.014505 -0.0017295; ! LNR is log of reliability;
ENDDATA

MAX = @SUM( COMPONENT(I): @SUM(UNITS(N):LNR(I,N)*X(I,N))) ;
@SUM( COMPONENT(I): @SUM(UNITS(N): WEIGHT(I)*N*X(I,N)) )<= WMAX;

@FOR (COMPONENT(I):
  @SUM (UNITS(N): X(I,N))=1;
@FOR (COMPONENT(I):
  @FOR (UNITS(N): @BIN (X(I,N)) ) );
```

## LINDO model:

```

MAX      - .35667 X( A, 1) - .094311 X( A, 2) - .027371 X( A, 3)
        - .22314 X( B, 1) - .040822 X( B, 2) - .0080322 X( B, 3)
        - .28768 X( C, 1) - .064539 X( C, 2) - .015748 X( C, 3)
        - .12783 X( D, 1) - .014505 X( D, 2) - .0017295 X( D, 3)

SUBJECT TO
2]  X( A, 1) + 2 X( A, 2) + 3 X( A, 3) + 2 X( B, 1) + 4 X( B, 2)
    + 6 X( B, 3) + X( C, 1) + 2 X( C, 2) + 3 X( C, 3) + 3 X( D, 1)
    + 6 X( D, 2) + 9 X( D, 3) <= 12
3]  X( A, 1) + X( A, 2) + X( A, 3) = 1
4]  X( B, 1) + X( B, 2) + X( B, 3) = 1
5]  X( C, 1) + X( C, 2) + X( C, 3) = 1
6]  X( D, 1) + X( D, 2) + X( D, 3) = 1

END
INTE    12

```

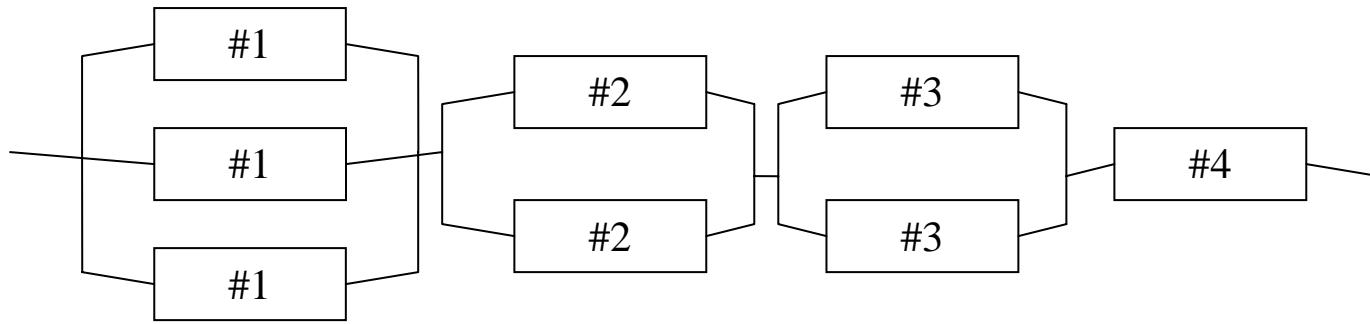
## Optimal Solution:

Objective value: - 0.2605620

Variable	Value	Reduced Cost
X( A, 3)	1.000000	0.2737100E-01
X( B, 2)	1.000000	0.4082200E-01
X( C, 2)	1.000000	0.6453900E-01
X( D, 1)	1.000000	0.1278300

*Note that*  $\exp\{-0.2605620\} = 0.77062$

which is in agreement with the dynamic programming solution.



## Optimal Design