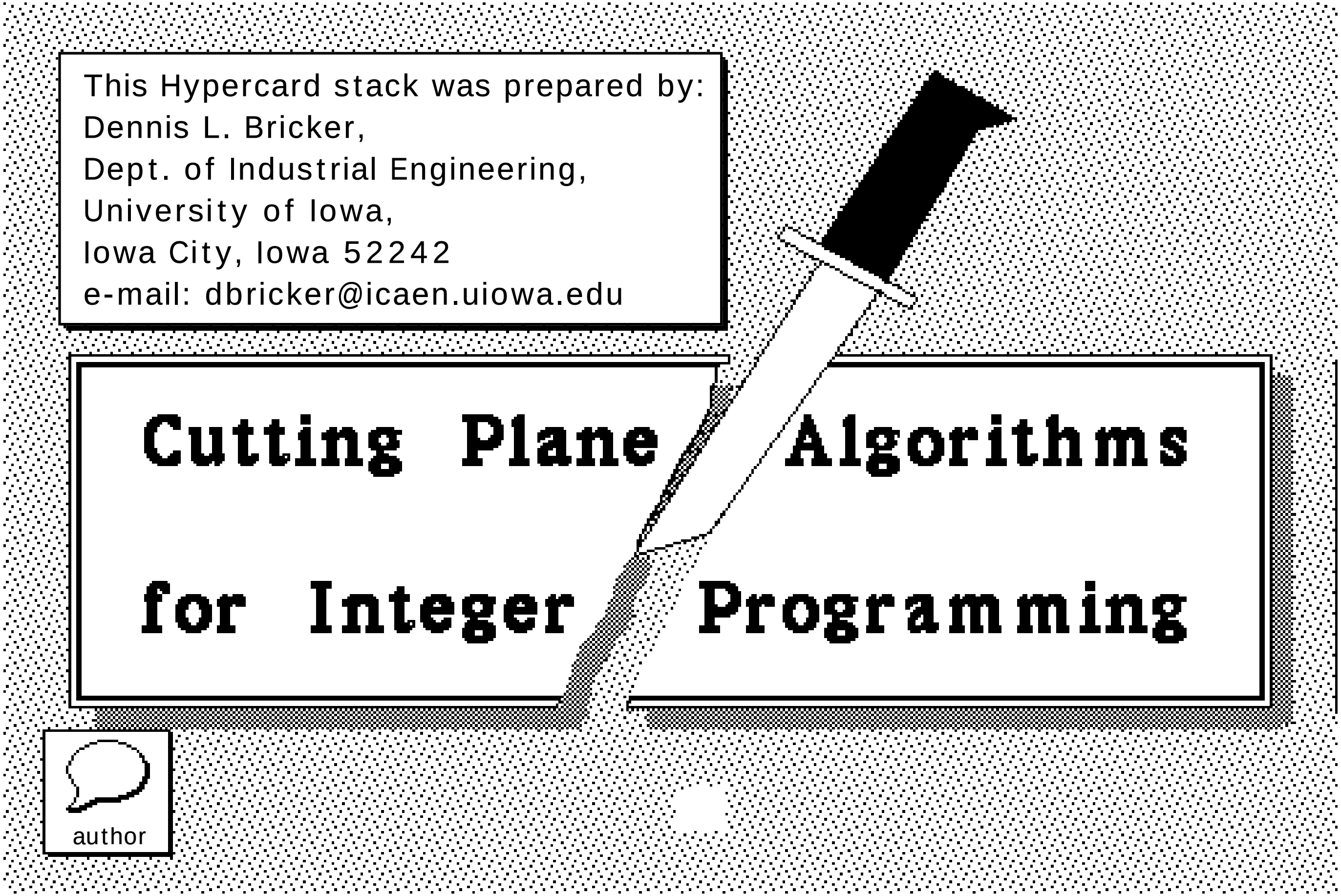

This Hypercard stack was prepared by:
Dennis L. Bricker,
Dept. of Industrial Engineering,
University of Iowa,
Iowa City, Iowa 52242
e-mail: dbricker@icaen.uiowa.edu
Cutting Plane Algorithms
for Integer Programming
author

Cutting-Plane Techniques: From a non-integer optimal solution of the LP relaxation, a constraint is derived and added to the LP, such that the LP solution is eliminated, but NO integer feasible solution is eliminated.

- ☞ Gomory's Fractional Cut
- Dual All-Integer Cut

Gomory's Fractional Cut

Suppose that the optimal LP tableau includes the row

$$\sum_{j=1}^{n} \alpha_{ij} x_j = \beta_i$$

Suppose that x_k is basic in this row, so that

$$x_k + \sum_{j \notin B} \alpha_{ij} x_j = \beta_i$$

where B = index set of basic variables.

Notation

$[\alpha_{ij}]$ = integer part of α_{ij}

f_{ij} = fractional part of
= $\alpha_{ij} - [\alpha_{ij}]$

$[\beta_i]$ = integer part of β_i

f_i = fractional part of
= $\beta_i - [\beta_i]$

Examples

$$\left[\frac{5}{4}\right] = 1 \qquad \left[\frac{3}{4}\right] = 0$$

$$\left[-\frac{3}{4}\right] = -1$$

Note that $[a] \le a$

$$x_k + \sum_{j \notin B} \alpha_{ij} x_j = \beta_i$$

may be written

$$x_k + \sum_{j \notin B} \left(\lfloor \alpha_{ij} \rfloor + f_{ij} \right) x_j = \lfloor \beta_i \rfloor + f_i$$

$$\Longrightarrow \quad x_k - \lfloor \beta_i \rfloor + \sum_{j \notin B} \lfloor \alpha_{ij} \rfloor x_j = f_i - \sum_{j \notin B} f_{ij} x_j$$

A NECESSARY condition for x_k & x_j $(j \notin B)$ to be integer is that the right-hand-side of

$$x_k - \lfloor \beta_i \rfloor + \sum_{j \notin B} \lfloor \alpha_{ij} \rfloor x_j = f_i - \sum_{j \notin B} f_{ij} x_j$$

is integer, i.e.,

$$f_i - \sum_{j \notin B} f_{ij} x_j \in \{\cdots -2, -1, 0, 1, 2, 3, \cdots\}$$

However, $f_i < 1$ & $f_{ij}x_j \geq 0$

imply that $f_i - \sum_{j \notin B} f_{ij}x_j < 1$

and, indeed, $f_i - \sum_{j \notin B} f_{ij}x_j$ must be no greater

than the largest integer < 1, i.e.,

$$f_i - \sum_{j \notin B} f_{ij}x_j \leq 0$$

Gomory's Fractional Cut

$$f_i - \sum_{j\notin B} f_{ij}x_j \le 0$$

$$\Longrightarrow \quad \sum_{j\notin B} f_{ij}x_j \ge f_i$$

$$- \sum_{j\notin B} f_{ij}x_j \le - f_i$$

$$- \sum_{j\notin B} f_{ij}x_j + \mathbf{S} = - f_i$$

slack variable

Gomory's Fractional Cut

$$- \sum_{j \notin B} f_{ij} x_j + S = - f_i$$

This constraint MUST be satisfied by all INTEGER feasible solutions of the source row!

However, it is NOT satisfied by the current LP solution if $f_i \neq 0$!

(Since $x_j = 0$ for $j \notin B$)

Gomory's Fractional Cut

$$\sum_{j \notin B} f_{ij}\, x_j \geq f_i$$

Example

basic variable → X_4 (source row)

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS
X_4	0	3	$\frac{1}{4}$	1	0	$-\frac{1}{3}$	$\frac{11}{4}$	$\frac{21}{5}$
	↓	↓	↓	↓	↓	↓	↓	↓
	0	0	$\frac{1}{4}$	0	0	$\frac{2}{3}$	$\frac{3}{4}$	$\geq \frac{1}{5}$

row of optimal LP tableau

$$\frac{1}{4}x_3 + \frac{2}{3}x_6 + \frac{3}{4}x_7 \geq \frac{1}{5}$$

Example

$$\frac{1}{4}x_3 + \frac{2}{3}x_6 + \frac{3}{4}x_7 \geq \frac{1}{5}$$

If x_3 , x_6 , *and* x_7 *are nonbasic in the current LP optimal tableau, then these variables are ZERO in the basic solution, and the above constraint is violated by the current LP optimal solution!*

Gomory's Cutting-Plane Algorithm

Step 0 Initialization

Solve the LP relaxation of the problem

Step 1 Optimality test

Is the LP solution integer? If so, stop.

Step 2 Cut

Choose a source row (with non-integer right-hand-side) and generate a cut.

Add cut to bottom of tableau

Step 3 Pivot

Re-optimize the LP, using the dual simplex algorithm.

Return to step 1.

All variables (including slack/surplus variables) must be integer.

If original inequality constraint has non-integer coefficients or right-hand-side, multiply both sides by an appropriate positive constant, e.g.

$$\frac{2}{5} x_1 + \frac{4}{3} x_2 \leq \frac{5}{2}$$

multiply both sides by 30

$$\Rightarrow 12\, x_1 + 40\, x_2 \leq 75$$

Choice of Source Row

Cuts may be generated using as source row:

- any row in optimal LP tableau which has a non-integer right-hand-side
- a multiple of any row in the LP tableau
- a linear combination of rows from the LP tableau

Choice of Source Row

While the strength of the cut varies, depending upon one's choice, no rule is known which will guarantee choosing the row yielding the strongest cut.

Heuristic rules

Choose, as source row, that which has

1) $\max_i \{f_i\}$

2) $\max_i \left\{ f_i \Big/ \sum_{j \notin B} f_{ij} \right\}$

3) $\min \left\{ \frac{1}{2} - f_i \right\}$

EXAMPLE

$$\text{Max } z = 2x_1 + x_2$$

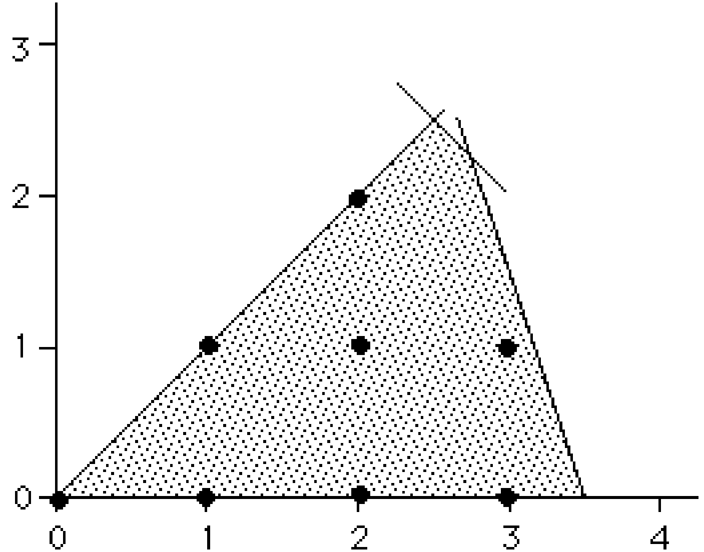
s.t.

$$x_1 + x_2 \le 5$$

$$-x_1 + x_2 \le 0$$

$$6x_1 + 2x_2 \le 21$$

$$x_1, x_2 \ge 0 \text{ \& integer}$$



EXAMPLE

Introduce slack variables to convert to equations:

$$
\begin{aligned}
\text{Max } & z = 2x_1 + x_2 \\
\text{subject to } & x_1 + x_2 + x_3 = 5 \\
& -x_1 + x_2 + x_4 = 0 \\
& 6x_1 + 2x_2 + x_5 = 21 \\
& x_j \in \{0,1,2,3,\cdots\}
\end{aligned}
$$

EXAMPLE

optimal LP tableau

$-z$	x_1	x_2	x_3	x_4	x_5	rhs
1	0	0	$-1/2$	0	$-1/4$	$-31/4$
0	1	0	$-1/2$	0	$1/4$	$11/4$
0	0	1	$3/2$	0	$-1/4$	$9/4$
0	0	0	-2	1	$1/2$	$1/2$

ANY of these rows could serve as the SOURCE row for a cut:

source row		cut
$x_1 - \frac{1}{2}x_3 + \frac{1}{4}x_5 = \frac{11}{4}$	$\Rightarrow$	$\frac{1}{2}x_3 + \frac{1}{4}x_5 \geq \frac{3}{4}$
$x_2 + \frac{3}{2}x_3 - \frac{1}{4}x_5 = \frac{0}{4}$	$\Rightarrow$	$\frac{1}{2}x_3 + \frac{3}{4}x_5 \geq \frac{1}{4}$
$-2x_3 + x_4 + \frac{1}{2}x_5 = \frac{1}{2}$	$\Rightarrow$	$\frac{1}{2}x_5 \geq \frac{1}{2}$

Graphical Representation of Cuts in X_1X_2-plane

substitute

$$\begin{cases} x_3 = 5 - x_1 - x_2 \\ x_5 = 21 - 6x_1 - 2x_2 \end{cases}$$

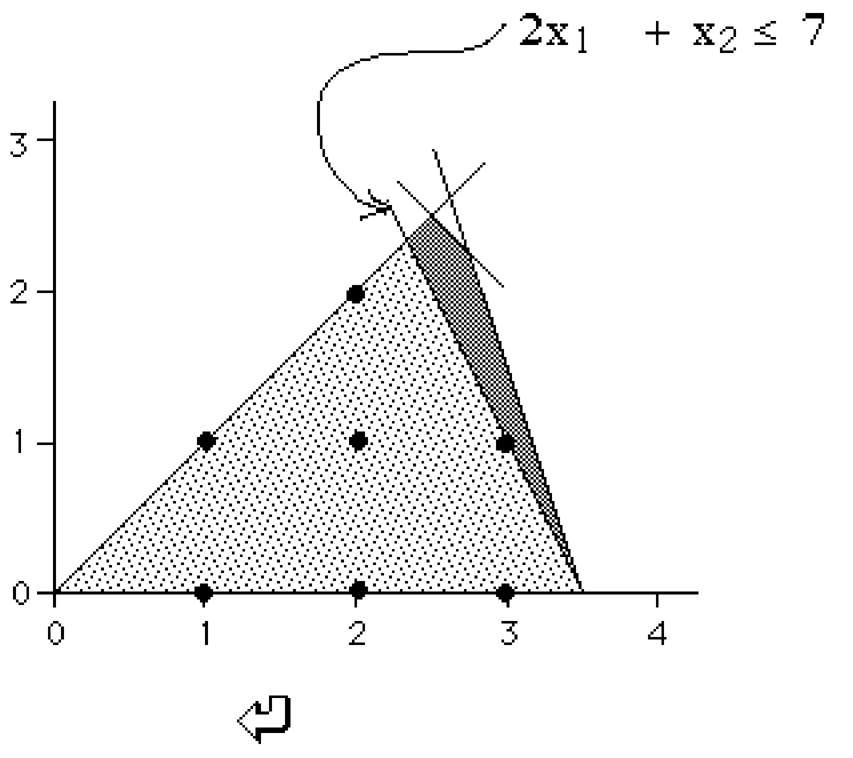
cut

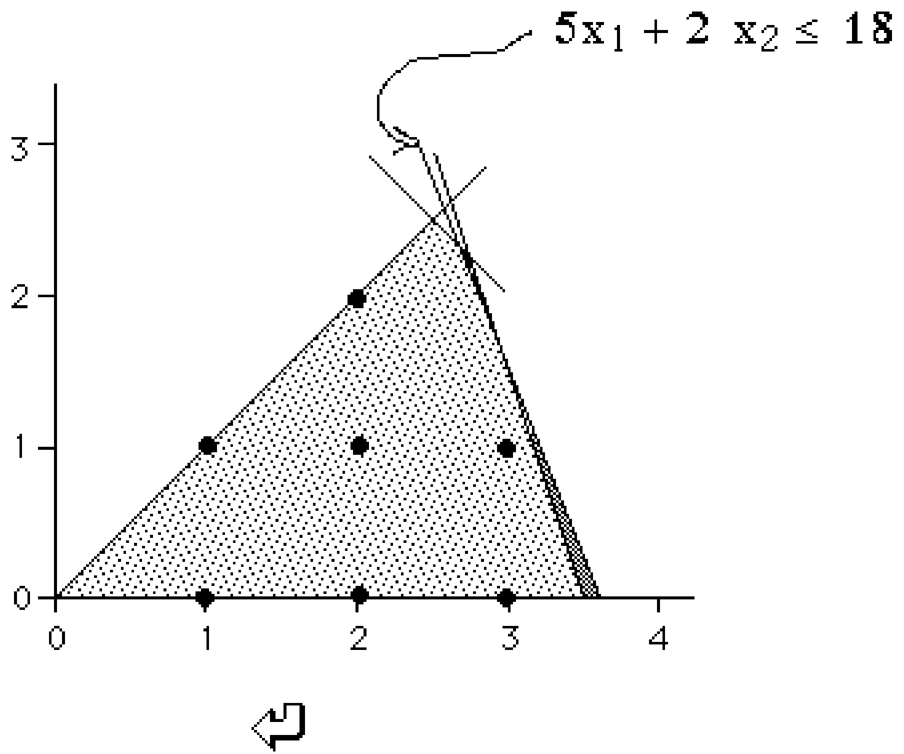
$$\frac{1}{2}x_3 + \frac{1}{4}x_5 \geq \frac{3}{4} \quad \Rightarrow \quad 2x_1 + x_2 \leq 7$$

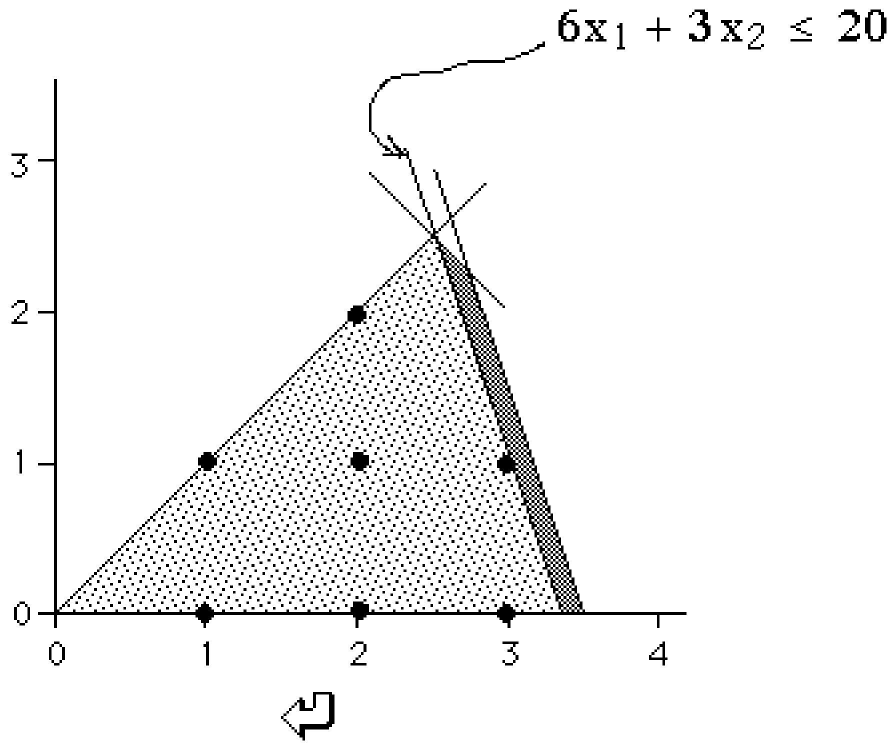
$$\frac{1}{2}x_3 + \frac{3}{4}x_5 \geq \frac{1}{4} \quad \Rightarrow \quad 5x_1 + 2x_2 \leq 18$$

$$\frac{1}{2}x_5 \geq \frac{1}{2} \quad \Rightarrow \quad 6x_1 + 3x_2 \leq 20$$

click mouse on cut to see effect


$2x_1 + x_2 \le 7$
3
2
1
0
0
1
2
3
4


$5x_1 + 2\ x_2 \leq\ 18$
3
2
1
0
0
1
2
3
4


6x1 + 3x2 ≤ 20
3
2
1
0
0
1
2
3
4

Dropping Cuts from Tableau

Each cut adds a new row & a new column (slack variable) to the tableau...

If ALL cuts are kept until the algorithm terminates, the tableau becomes so large as to be "unwieldy"!

When a cut is no longer "useful", it would be advantageous to be able to delete that cut.

Dropping Cuts from Tableau

When a cut is added to the tableau, & the dual simplex pivot removes its slack variable from the basis, the cut is a "tight" constraint, i.e., its slack variable is zero.

If a cut's slack variable re-enters the basis at a later iteration, then the cut has become inactive and may then be dropped from the tableau.

EXAMPLE

$$\text{Max } z = 2x_1 + x_2$$

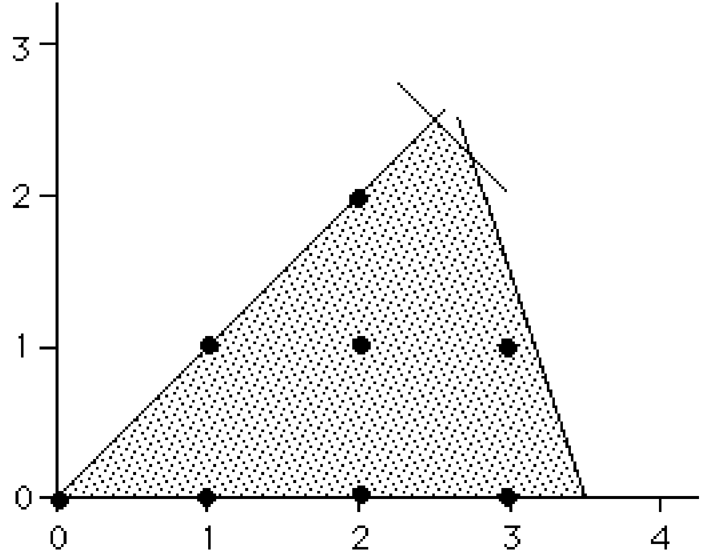
s.t.

$$x_1 + x_2 \le 5$$

$$-x_1 + x_2 \le 0$$

$$6x_1 + 2x_2 \le 21$$

$$x_1, x_2 \ge 0 \text{ \& integer}$$



Initial Optimal LP tableau

Current LP Tableau

z	1	2	3	4	5	B
1	0	0	−0.5	0	−0.25	−7.75
0	0	1	1.5	0	−0.25	2.25
0	0	0	−2	1	0.5	0.5
0	1	0	−0.5	0	0.25	2.75

Variables:

(Negative of) objective function value: z

Original structural variables: 1 2

Original slack/surplus variables: 3 4 5

Slack variables for cuts:

The rows having non-integer right-hand-side are 2 3 4

Source row is # 2

i	2	3	5	6	rhs
Source row	1	1.5	¯0.25	0	2.25
Cut	0	¯0.5	¯0.75	1	¯0.25

(X[6] (= slack variable for new cut) is basic but < 0)

The cut which is added is (in terms of original variables):

1	2		b
5	2	≤	18

Current LP Tableau

z	1	2	3	4	5	6	B
1	0	0	¯0.5	0	¯0.25	0	¯7.75
0	0	1	1.5	0	¯0.25	0	2.25
0	0	0	¯2	1	0.5	0	0.5
0	1	0	¯0.5	0	0.25	0	2.75
0	0	0	¯0.5	0	¯0.75	1	¯0.25 ← cut

Variables:
(Negative of) objective function value: z
Original structural variables: 1 2
Original slack/surplus variables: 3 4 5
Slack variables for cuts: 6

Tableau is now primal infeasible (but dual feasible!)

Solving current LP

Performing dual simplex pivot in row 5

Potential pivot columns: X[3 5]

i	3	5
Rel. Profit	¯0.5	¯0.25
Subs. rate	¯0.5	¯0.75
Ratio	1	0.333

Minimum ratio is in column 5,
which is selected as pivot column

Current LP Tableau

z	1	2	3	4	5	6	B
1	0	0	¯0.5	0	¯0.25	0	¯7.75
0	0	1	1.5	0	¯0.25	0	2.25
0	0	0	¯2	1	0.5	0	0.5
0	1	0	¯0.5	0	0.25	0	2.75
0	0	0	¯0.5	0	¯0.75	1	¯0.25

↑ pivot column

Current LP Tableau

z	1	2	3	4	5	6	B
1	0	0	$^{-}$0.333	0	0	$^{-}$0.333	$^{-}$7.67
0	0	1	1.67	0	0	$^{-}$0.333	2.33
0	0	0	$^{-}$2.33	1	0	0.667	0.333
0	1	0	$^{-}$0.667	0	0	0.333	2.67
0	0	0	0.667	0	1	$^{-}$1.33	0.333

Variables:
(Negative of) objective function value: z
Original structural variables: 1 2
Original slack/surplus variables: 3 4 5
Slack variables for cuts: 6

The rows having non-integer right-hand-side are 2 3 4 5

Source row is # 2

i	2	3	6	7	rhs
Source row	1	1.67	¯0.333	0	2.33
Cut:	0	¯0.667	¯0.667	1	¯0.333

(X[7] (= slack variable for new cut) is basic but < 0)

The cut which is added is (in terms of original variables):

1	2		b
4	2	≤	15

Current LP Tableau

z	1	2	3	4	5	6	7	B
1	0	0	¯0.333	0	0	¯0.333	0	¯7.67
0	0	1	1.67	0	0	¯0.333	0	2.33
0	0	0	¯2.33	1	0	0.667	0	0.333
0	1	0	¯0.667	0	0	0.333	0	2.67
0	0	0	0.667	0	1	¯1.33	0	0.333
0	0	0	¯0.667	0	0	¯0.667	1	¯0.333 ← cut

Variables:
- (Negative of) objective function value: z
- Original structural variables: 1 2
- Original slack/surplus variables: 3 4 5
- Slack variables for cuts: 6 7

Solving current LP

Performing dual simplex pivot in row 6

Potential pivot columns: X[3 6]

i	3	6
Rel. Profit	¯0.333	¯0.333
Subs. rate	¯0.667	¯0.667
Ratio	0.5	0.5

Minimum ratio is in column 3,
which is selected as pivot column

Current LP Tableau

z	1	2	3	4	5	6	7	B
1	0	0	-0.333	0	0	-0.333	0	-7.67
0	0	1	1.67	0	0	-0.333	0	2.33
0	0	0	-2.33	1	0	0.667	0	0.333
0	1	0	-0.667	0	0	0.333	0	2.67
0	0	0	-0.667	0	1	-1.33	0	0.333
0	0	0	-0.667	0	0	-0.667	1	-0.333

pivot column

Current LP Tableau

z	1	2	3	4	5	6	7	B
1	0	0	0	0	0	0	⁻0.5	⁻7.5
0	0	1	0	0	0	⁻2	2.5	1.5
0	0	0	0	1	0	3	⁻3.5	1.5
0	1	0	0	0	0	1	⁻1	3
0	0	0	0	0	1	⁻2	1	0
0	0	0	1	0	0	1	⁻1.5	0.5

Variables:

(Negative of) objective function value: z
Original structural variables: 1 2
Original slack/surplus variables: 3 4 5
Slack variables for cuts: 6 7

The rows having non-integer right-hand-side are 2 3 6

Source row is # 2

i	2	6	7	8	rhs
Source row	1	¯2	2.5	0	1.5
Cut:	0	0	¯0.5	1	¯0.5

(X[8] (= slack variable for new cut) is basic but < 0)

The cut which is added is (in terms of original variables):

1	2		b
2	1	≤	7

Current LP Tableau

z	1	2	3	4	5	6	7	8	B
1	0	0	0	0	0	0	-0.5	0	-7.5
0	0	1	0	0	0	-2	2.5	0	1.5
0	0	0	0	1	0	3	-3.5	0	1.5
0	1	0	0	0	0	1	-1	0	3
0	0	0	0	0	1	-2	1	0	0
0	0	0	1	0	0	1	-1.5	0	0.5
0	0	0	0	0	0	0	-0.5	1	-0.5 ← cut

Variables:
(Negative of) objective function value: z
Original structural variables: 1 2
Original slack/surplus variables: 3 4 5
Slack variables for cuts: 6 7 8

Solving current LP

Performing dual simplex pivot in row 7

Potential pivot columns: X[7]

i:	7
Rel. Profit	¯0.5
Subs. rate	¯0.5
Ratio	1

Minimum ratio is in column 7,
which is selected as pivot column

Resulting solution is again infeasible (variable < 0)

Current LP Tableau

z	1	2	3	4	5	6	7	8	B
1	0	0	0	0	0	0	−0.5	0	−7.5
0	0	1	0	0	0	−2	2.5	0	1.5
0	0	0	0	1	0	3	−3.5	0	1.5
0	1	0	0	0	0	1	−1	0	3
0	0	0	0	0	1	−2	1	0	0
0	0	0	1	0	0	1	−1.5	0	0.5
0	0	0	0	0	0	0	−0.5	1	−0.5

z	1	2	3	4	5	6	7	8	B
1	0	0	0	0	0	0	0	⁻1	⁻7
0	0	1	0	0	0	⁻2	0	5	⁻1
0	0	0	0	1	0	3	0	⁻7	5
0	1	0	0	0	0	1	0	⁻2	4
0	0	0	0	0	1	⁻2	0	2	⁻1
0	0	0	1	0	0	1	0	⁻3	2
0	0	0	0	0	0	0	1	⁻2	1

As a result of the previous dual simplex pivot, the right-hand-side of the new row becomes positive, but further dual simplex pivots are necessary, because negative numbers have appeared in other rows!

z	1	2	3	4	5	6	7	8	B
1	0	0	0	0	0	0	0	⁻1	⁻7
0	0	1	0	0	0	⁻2	0	5	⁻1
0	0	0	0	1	0	3	0	⁻7	5
0	1	0	0	0	0	1	0	⁻2	4
0	0	0	0	0	1	⁻2	0	2	⁻1
0	0	0	1	0	0	1	0	⁻3	2
0	0	0	0	0	0	0	1	⁻2	1

Next pivot row should be either row 2 or row 5.

Performing dual simplex pivot in row 2

Potential pivot columns: X[6]

i	6
Rel. Profit	0
Subs. rate	⁻2
Ratio	0

Minimum ratio is in column 6,
which is selected as pivot column

z	1	2	3	4	5	6	7	8	B
1	0	0	0	0	0	0	0	-1	-7
0	0	1	0	0	0	-2	0	5	-1
0	0	0	0	1	0	3	0	-7	5
0	1	0	0	0	0	1	0	-2	4
0	0	0	0	0	1	-2	0	2	-1
0	0	0	1	0	0	1	0	-3	2
0	0	0	0	0	0	0	1	-2	1

Current LP Tableau

z	1	2	3	4	5	6	7	8	B
1	0	0	0	0	0	0	0	⁻1	⁻7
0	0	⁻0.5	0	0	0	1	0	⁻2.5	0.5
0	0	1.5	0	1	0	0	0	0.5	3.5
0	1	0.5	0	0	0	0	0	0.5	3.5
0	0	⁻1	0	0	1	0	0	⁻3	0
0	0	0.5	1	0	0	0	0	⁻0.5	1.5
0	0	0	0	0	0	0	1	⁻2	1

Variables:
(Negative of) objective function value: z
Original structural variables: 1 2
Original slack/surplus variables: 3 4 5
Slack variables for cuts: 6 7 8

The rows having non-integer right-hand-side are 2 3 4 6

From which row do you wish to generate the cut?
⎕:
2

The cut which is added is (in terms of original variables):

1	2		b
1	0	≤	3

Source row is # 2

i:	2	6	8	9	rhs
Source row:	¯0.5	1	¯2.5	0	0.5
Cut:	¯0.5	0	¯0.5	1	¯0.5

(X[9] (= slack variable for new cut) is basic but < 0)

Current LP Tableau

z	1	2	3	4	5	6	7	8	9	B
1	0	0	0	0	0	0	0	⁻1	0	⁻7
0	0	⁻0.5	0	0	0	1	0	⁻2.5	0	0.5
0	0	1.5	0	1	0	0	0	0.5	0	3.5
0	1	0.5	0	0	0	0	0	0.5	0	3.5
0	0	⁻1	0	0	1	0	0	⁻3	0	0
0	0	0.5	1	0	0	0	0	⁻0.5	0	1.5
0	0	0	0	0	0	0	1	⁻2	0	1
0	0	⁻0.5	0	0	0	0	0	⁻0.5	1	⁻0.5

Variables:
- (Negative of) objective function value: z
- Original structural variables: 1 2
- Original slack/surplus variables: 3 4 5
- Slack variables for cuts: 6 7 8 9

Solving current LP

Performing dual simplex pivot in row 8

Potential pivot columns: X[2 8]

i:	2	8
Rel. Profit	0	¯1
Subs. rate	¯0.5	¯0.5
Ratio	0	2

Minimum ratio is in column 2,
which is selected as pivot column

z	1	2	3	4	5	6	7	8	9	B
1	0	0	0	0	0	0	0	−1	0	−7
0	0	−0.5	0	0	0	1	0	−2.5	0	0.5
0	0	1.5	0	1	0	0	0	0.5	0	3.5
0	1	0.5	0	0	0	0	0	0.5	0	3.5
0	0	−1	0	0	1	0	0	−3	0	0
0	0	0.5	1	0	0	0	0	−0.5	0	1.5
0	0	0	0	0	0	0	1	−2	0	1
0	0	−0.5	0	0	0	0	0	−0.5	1	−0.5

Current LP Tableau

z	1	2	3	4	5	6	7	8	9	B
1	0	0	0	0	0	0	0	⁻1	0	⁻7
0	0	0	0	0	0	1	0	⁻2	⁻1	1
0	0	0	0	1	0	0	0	⁻1	3	2
0	1	0	0	0	0	0	0	0	1	3
0	0	0	0	0	1	0	0	⁻2	⁻2	1
0	0	0	1	0	0	0	0	⁻1	1	1
0	0	0	0	0	0	0	1	⁻2	0	1
0	0	1	0	0	0	0	0	1	⁻2	1

All variables are integer!

Variables:

(Negative of) objective function value: z
Original structural variables: 1 2
Original slack/surplus variables: 3 4 5
Slack variables for cuts: 6 7 8 9

Current List of Cuts

#	1	2		b
1)	5	2	≤	18
2)	4	2	≤	15
3)	2	1	≤	7
4)	1	0	≤	3

Optimal Solution:

$X_1 = 3$

$X_2 = 1$