Convexity of Sets & Functions

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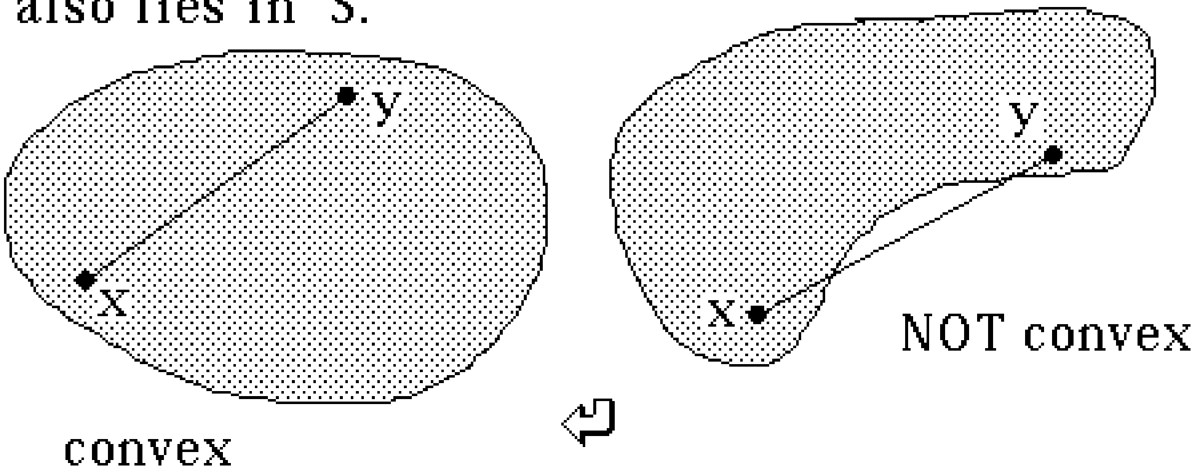
The property of "CONVEXITY" of sets and of functions is central to most approaches to nonlinear programming.

- ☞ Convex Set
- ☞ Convex Function

The more "unified" approach first defines convexity of sets, and bases the definition of convex function upon convexity of sets.

Convex Set

A set S is CONVEX if for every pair of elements x and y in S, the line segment joining x and y also lies in S.

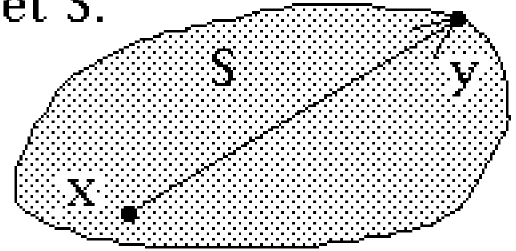


This means that, if we are solving the problem

$$\boxed{\begin{array}{l}\text{Minimize } f(x)\\ \text{subject to } x \in S\end{array}}$$

and S is convex, that there is a linear path such that we can follow this path directly from the starting point $x \in S$ to the optimal solution $y \in S$ without leaving the set S.

Unfortunately, we are seldom able to determine this line!



The ***line segment*** between x and y is given by

$$\lambda y + (1-\lambda)x = x + \lambda(y-x) \qquad \textbf{for} \quad \lambda \in [0,1]$$

where

$$\lambda=0 \implies \lambda y + (1-\lambda)x = x$$

$$\lambda=1 \implies \lambda y + (1-\lambda)x = y$$

$$\lambda=1/2 \implies \lambda y + (1-\lambda)x = 1/2(x+y)$$ *(midpt. of segment)*,

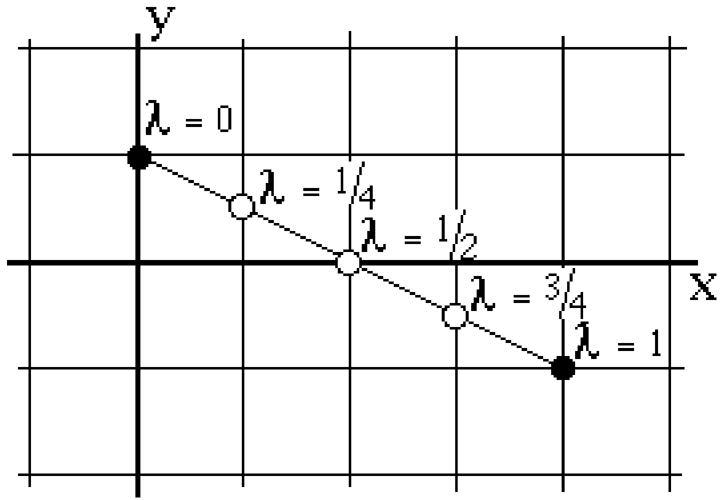
etc.

Example

line segment

Let x=(0,1) and y=(4, −1) be points in the plane.

λ	$\lambda y + (1-\lambda)x$
0	(0,1)
0.25	(1, 0.5)
0.50	(2, 0)
0.75	(3, −0.5)
1	(4, −1)



Convex Combination

If $x^{(1)}, x^{(2)}, \ldots x^{(k)}$ are vectors in R^n, and

$\lambda_1, \lambda_2, \cdots \lambda_k$ are nonnegative numbers whose

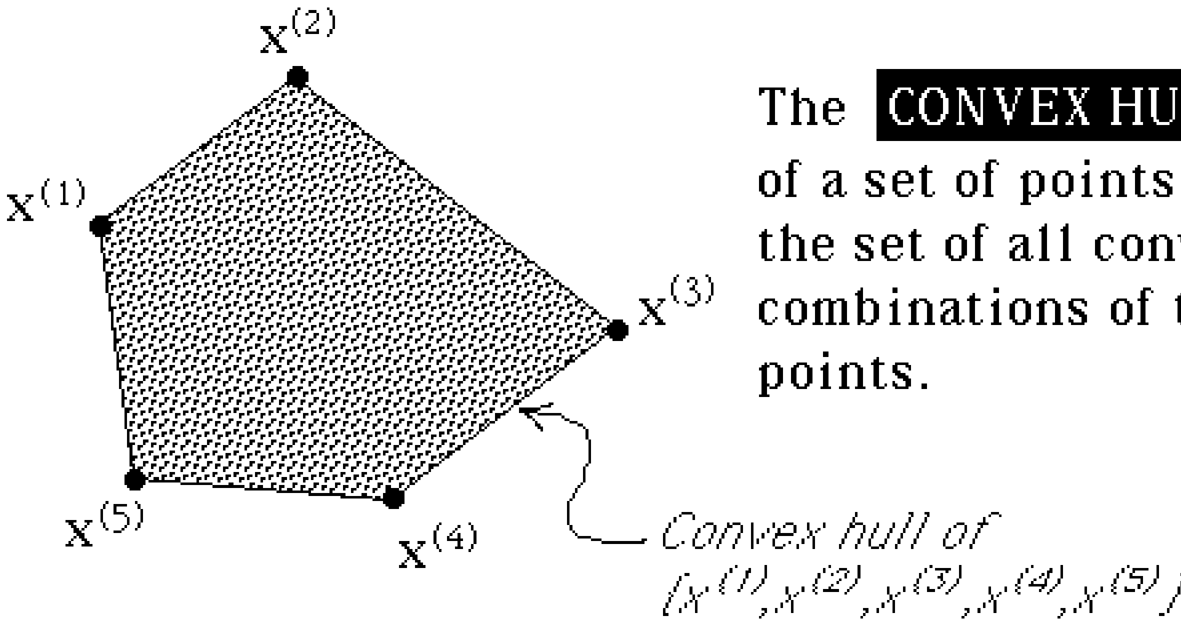
sum is 1, i.e., $\sum_{i=1}^{k} \lambda_i = 1$

then $\lambda_1 x^{(1)} + \lambda_2 x^{(2)} + \cdots \lambda_k x^{(k)}$

is a convex combination (weighted average) of

$x^{(1)}, x^{(2)}, \ldots x^{(k)}$

In particular, a point on a line segment is a convex combination of the endpts. of the segment!



The **CONVEX HULL** of a set of points is the set of all convex combinations of those points.

Theorem If $x^{(1)}, x^{(2)}, \ldots x^{(k)} \in S$ where

S is a convex set, then every convex combination of the points $x^{(1)}, x^{(2)}, \ldots x^{(k)}$ is an element of S.

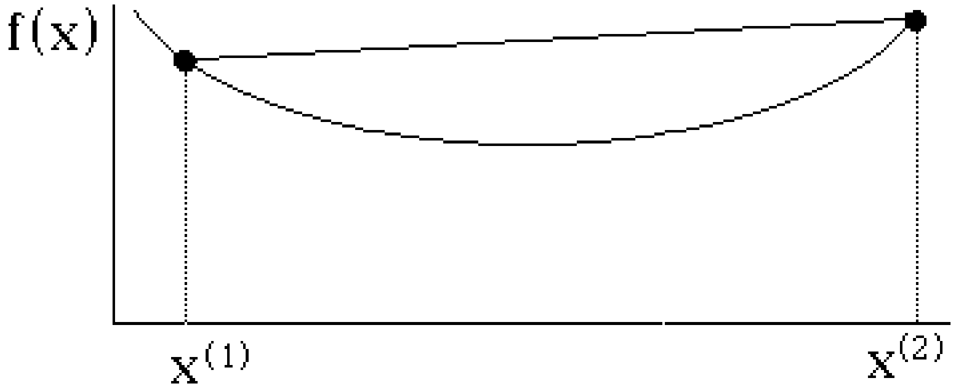
That is, if S is convex then S equals its convex hull.

Convex Function

A function f(x) is ***convex*** if:

$$f\left(\lambda x^{(1)} + (1-\lambda)x^{(2)}\right) \le \lambda f\left(x^{(1)}\right) + (1-\lambda)\, f\left(x^{(2)}\right) \quad \forall\, \lambda \in [0,1]$$

For example, f evaluated at the midpoint of two points is less than the average of the function values at the two points.

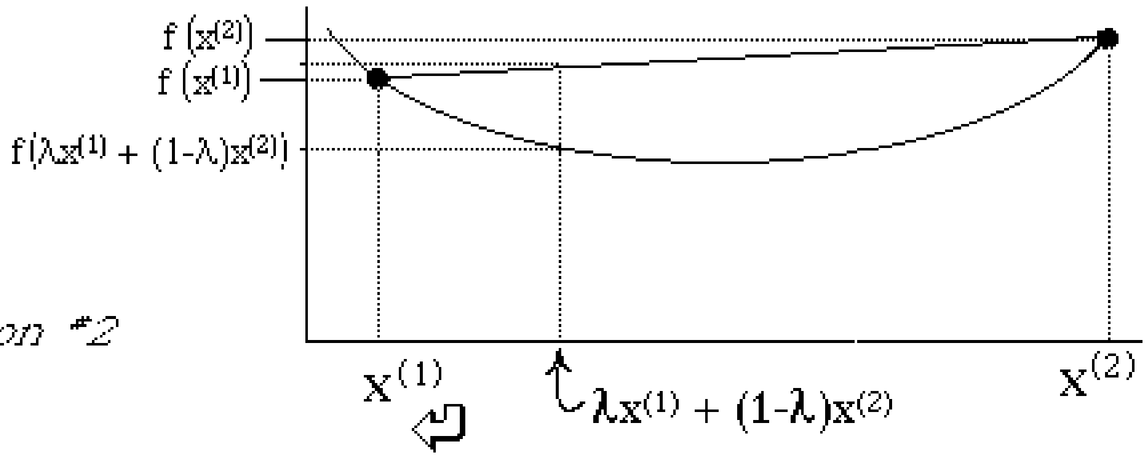


Definition #1

Convex Function

A function f(x) is ***convex*** if:

$$f\left(\lambda x^{(1)} + (1-\lambda)x^{(2)}\right) \le \lambda f\left(x^{(1)}\right) + (1-\lambda)\, f\left(x^{(2)}\right) \quad \forall \lambda \in [0,1]$$

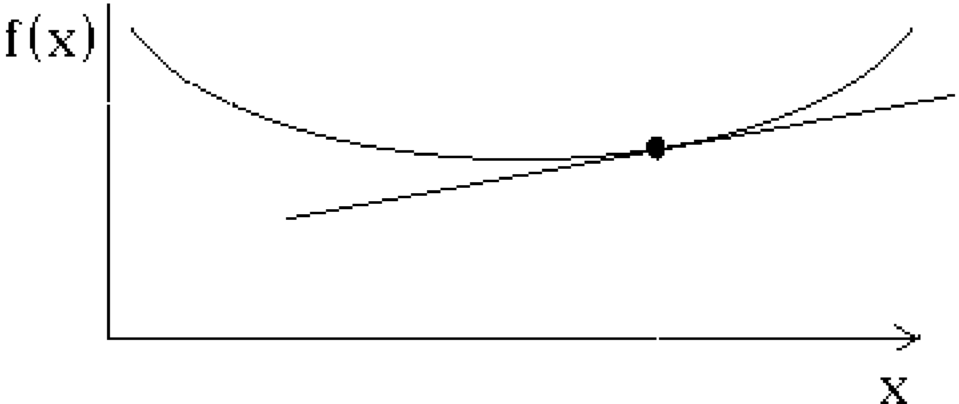


Definition #2

Convex Function

A differentiable function f(x) is *convex* if:

the tangent line (hyperplane) to the graph lies on or below the graph:

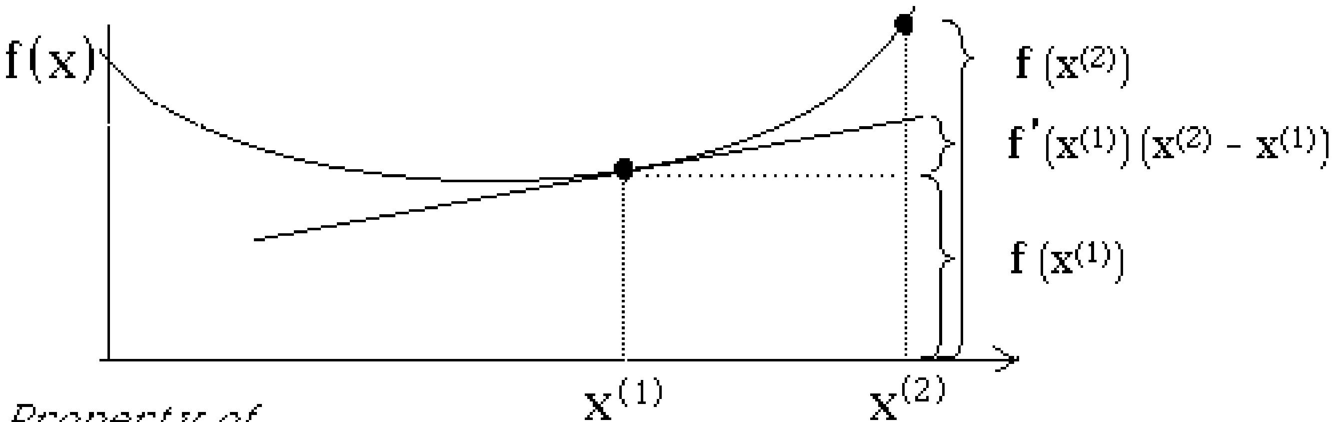


Property of convex function

Convex Function

A differentiable function f(x) is ***convex*** if:

$$f\left(x^{(1)}\right) + f'\left(x^{(1)}\right)\left(x^{(2)} - x^{(1)}\right) \le f\left(x^{(2)}\right)$$

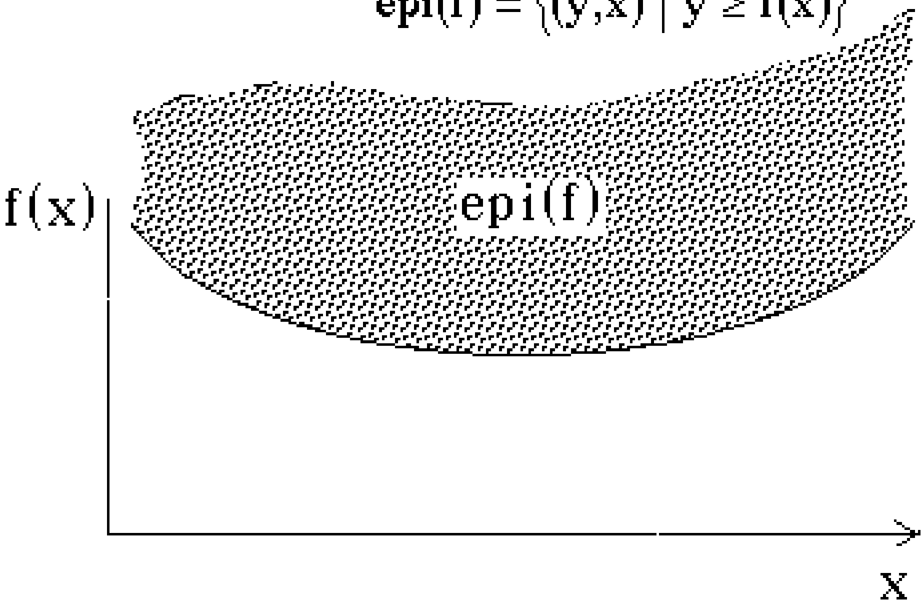


Property of convex function

Epigraph

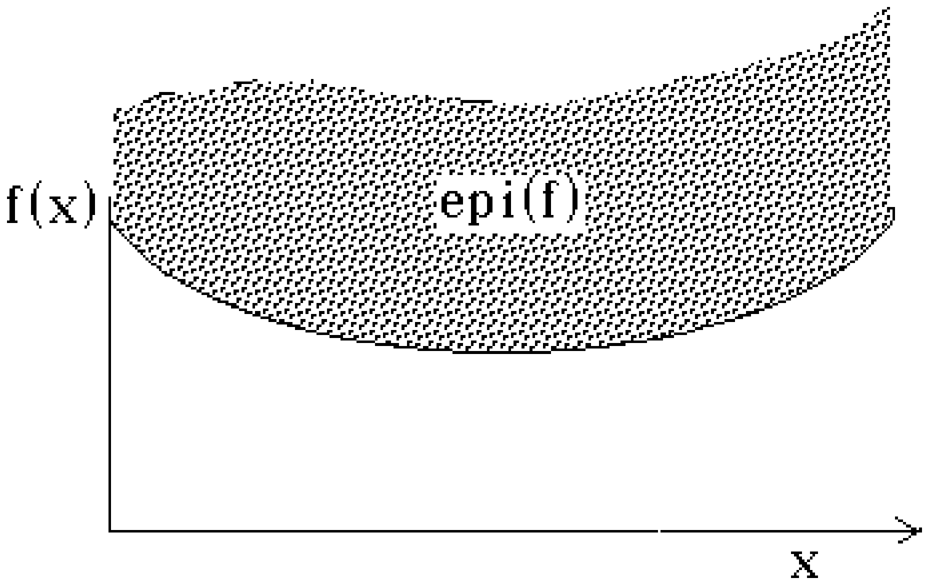
The epigraph of a function is the set

$$\mathbf{epi(f)} = \left\{(\mathbf{y},\mathbf{x}) \mid \mathbf{y} \geq \mathbf{f(x)}\right\}$$



Convex Function

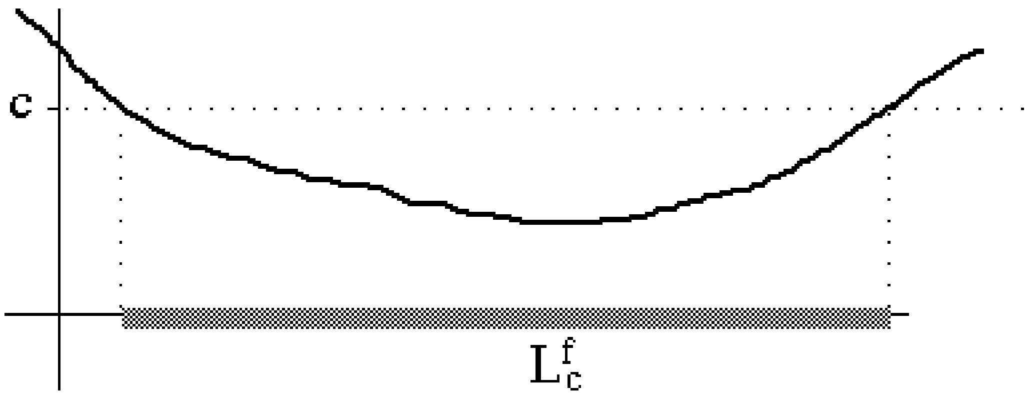
A function f(x) is ***convex*** if the set **epi(f)** is convex



relationship between the concepts of convexity of a set and of a function!

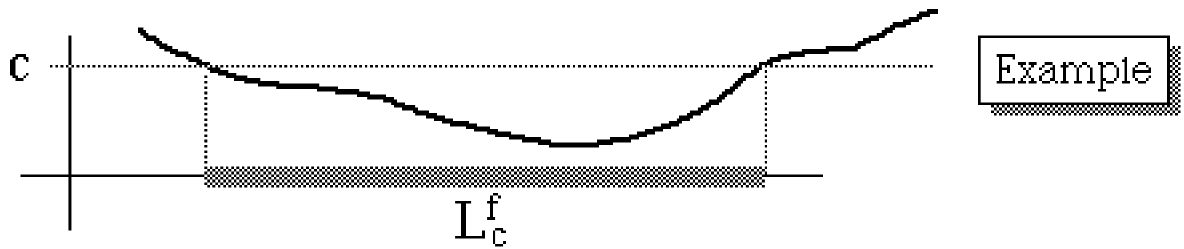
For any real value c, the **Level Set** of the function f is the set

$$L_c^f = \{x \mid f(x) \leq c\}$$



If f is a convex function, then L_c^f is convex.

However, the convexity of the level sets does NOT imply convexity of the function.

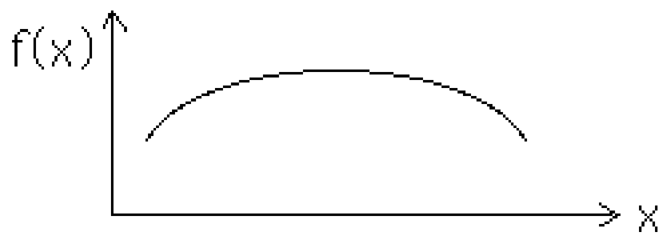


If all the level sets of a function are convex, then the function is *quasi* convex.

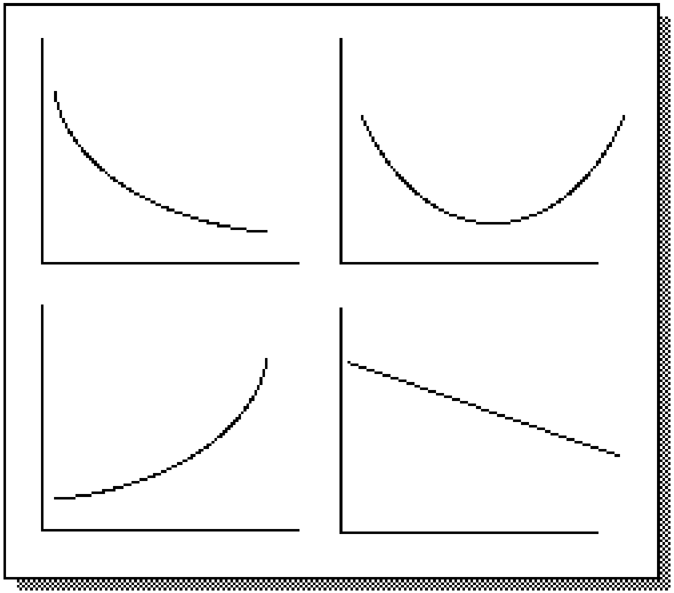
Concave Function

A function f is ***concave*** if

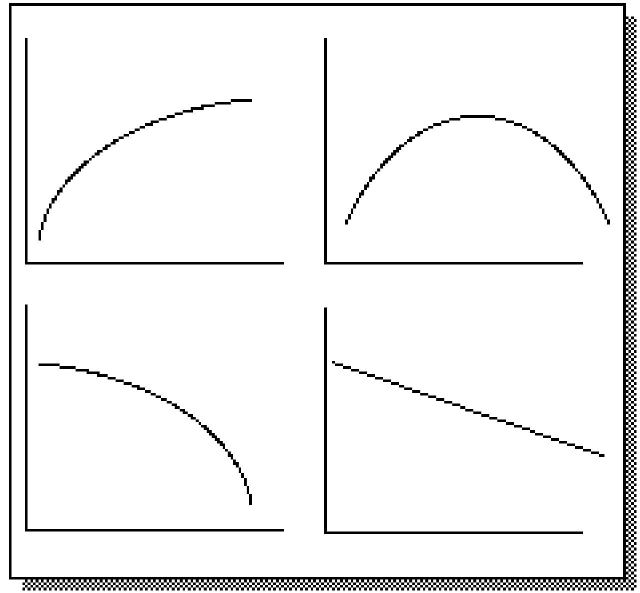
- its negative, $(-f)$, is convex
- a chord between 2 points on the graph lies on or below the graph
- a tangent line (hyperplane) to the graph lies on or above the graph
- the hypergraph $\{(y,x) \mid y \leq f(x)\}$ is convex



Examples



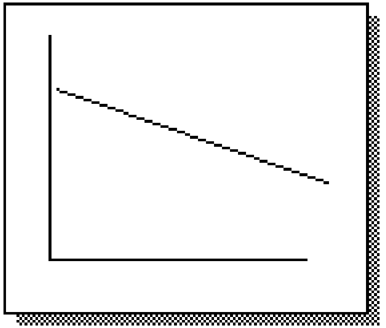
convex functions



concave functions

Examples

A linear function is *both* convex and concave!



A function may be neither convex nor concave: