56:271 Nonlinear Programming
Final Examination - May 1991

Answer ALL of Parts A & B, and 3 problems from Part C.

pppp *PART A* pppp

Indicate whether each statement is *true* (+) or *false* (o). **If false, state why** or **give a counterexample.**

_____ 1. If the current x lies on the boundary of the feasible region, the gradient projection method computes a search direction by projecting the steepest descent method onto the boundary.
_____ 2. All posynomials are convex functions.
_____ 3. All quadratic functions are convex.
_____ 4. The function $(x^2 + 2x)(e^y + y)$ is separable.
_____ 5. Box's COMPLEX method does not make use of derivatives.
_____ 6. ^2f denotes the Hessian matrix of the function f.
_____ 7. The function (5x - 2y) is a convex function of (x,y).
_____ 8. The sum of two convex functions is convex.
_____ 9. The product of two convex functions is convex.
_____10. If f is differentiable, then $f(x^*) = 0$ is a necessary condition for optimality of x^* in the problem : Min f(x).
_____11. The geometric programming dual problem could be solved using the GRG algorithm.
_____12. The Jacobian matrix of the necessary conditions for optimality of the problem "minimize f(x)" is the Hessian matrix of f(x).
_____13. The steepest descent algorithm requires the calculation of partial derivatives.
_____14. A function which is not convex is called "concave".
_____15. The "independent" variables of the GRG algorithm are essentially the same as the "basic" variables of the LP simplex algorithm.
_____16. A differentiable function is convex if and only if its Hessian matrix is non-negative.
_____17. A positive definite matrix always has all positive elements, although a matrix whose elements are positive need not be positive definite.
_____18. The function f(x,y) = xy is a concave function of x and y.
_____19. x ln x is a convex function of x (x>0).
_____20. If the GP dual problem is infeasible, then the primal GP problem's objective function is unbounded below.
_____21. If $f_i(x)$ is convex for i=1,2,...n, then the function $\Phi(x)$ defined by $\Phi(x) = \min\{f_i(x): i=1,2,...n\}$ is a convex function.
_____22. Each iteration of Fibonacci's procedure eliminates more of the interval of uncertainty than does an iteration of the Golden Section Search procedure.
_____23. The Quadratic Interpolation procedure for one-dimensional search does not require the computation of derivatives.
_____24. Newton's algorithm is guaranteed to find the unconstrained minimum of a quadratic function of 3 variables in a single iteration.
_____25. The Fletcher-Powell algorithm will find the unconstrained minimum of a quadratic function of 3 variables in no more than 3 iterations.
_____26. The GRG algorithm requires that dependent variables <u>not</u> be at either their upper or lower bounds.
_____27. A matrix is negative definite if & only if all its eigenvalues are negative.

_____28. Cubic interpolation procedure for one-dimensional minimization requires the computation of second derivatives.

_____29. "GRG" means "Gradient Restriction Generation"

_____30. A linear function is neither convex nor concave.

_____31. ^2f is the Hessian matrix of the function f.

_____32. The Jacobian matrix of the equation $f(x) = 0$ is called the Hessian matrix.

_____33. The GRG algorithm requires the use of a one-dimensional search method.

_____34. A twice-differentiable function is convex if and only if its Hessian matrix is nonnegative everywhere.

_____35. The function e^{ax} is convex for all values of a.

_____36. Any two orthogonal directions are conjugate with respect to the identity matrix.

_____37. The Fletcher-Reeves method is also known as a "Quasi-Newton" method.

_____38. A positive definite matrix always has all positive elements, although a matrix whose elements are positive need not be positive definite.

_____39. The Lagrangian dual of a convex quadratic programming problem is a quadratic programming problem.

_____40. Powell's algorithm does not require the computation of partial derivatives.

_____41. The Fletcher-Powell method is also known as the "Conjugate Gradient" method.

_____42. In the "Golden Section Search" method, one-third of the interval of uncertainty is eliminated at a typical iteration.

_____43. If all posynomials in a (posynomial) GP problem are condensed into single terms, then the resulting problem has a feasible region which includes the original feasible region.

_____44. If all posynomials in a (posynomial) GP problem are condensed into single terms, then the resulting problem may be transformed into an LP problem.

_____45. The gradient projection method computes Lagrange multipliers at each iteration, and stops when they have the appropriate sign.

_____46. "Quasi-Newton" search methods for unconstrained minimization require the computation of second partial derivatives.

pppp *PART B* pppp

1. According to the Arithmetic-Geometric Mean Inequality, the average of the numbers 3 and 12 exceeds _ _ _ _ _ _ _.

2. The expression $({}^c/_w)^w$ evaluated at w=0 yields _ _ _ _ _ _ _.

3. I wish to solve the equation x ln x = 1. My initial "guess" is x=1. What is the next approximation to the solution provided by the Newton-Raphson method? _ _ _ _ _ _ _ _.

4. If possible, give an example of an optimal solution of a nonlinear programming problem which does not satisfy the Kuhn-Tucker conditions for optimality.

5. It sometimes happens that you are able to find an optimal solution to the GP dual problem, but you are then unable to compute the optimal values of the primal variables. Explain why this can happen. What can be done to then compute the primal variables?

6. Orthogonal directions are conjugate with respect to what matrix?

7. Explain why, when solving a geometric programming problem, the problem may be made more difficult if the value of one of the variables is fixed.

8. Explain why, if a primal GP constraint is not "tight" at the optimum, then the GP dual objective function , in its separable form, is not differentiable at the optimum.

pppp *PART C* pppp

1. Consider the function $f(x) = x \ln x$, $x>0$

   a. Is f a convex function? a concave function?

   b. Even though $(\ln x)$ is not defined at $x=0$, $x \ln x$ converges as $x \to 0^+$. The function f may be defined to be what value at $x=0$ so that it is a continuous function?

   c. What is $f'(x)$ for $x>0$? $f'(0)$?

   d. Suppose that I wish to find x such that $f(x)=1$. Describe an iterative procedure to do this, and illustrate one step, starting at $x^o = 1$.

   e. Suppose that I wish to find x (>0) minimizing the function f. Describe an iterative procedure to do this, and illustrate one step, starting at $x^o = 1$.

2. **Lagrangian Duality**: Consider the two problems

   A: Minimize $-x^2$ subject to $(x-1)^2 \le 1$ B: Minimize x^2 subject to $(x-1)^2 \le 1$

   (note: in both A & B, no nonnegativity restrictions!)

For each problem (A) and (B) above:

   a. Sketch a very rough graph and indicate the optimal solution (x^*).

   b. Write the Kuhn-Tucker conditions for optimality.

   c. Check whether the optimum indicated in (a) satisfies the Kuhn-Tucker conditions.

   d. State the objective function of the Lagrangian dual problem (as a function of the Lagrange multiplier λ alone.) Is this function convex, concave, both, or neither?

   e. State the Lagrangian dual problem.

   f. Solve the Lagrangian dual problem.

   g. Is there a duality gap for this problem?

3. **Quadratic Forms.** Consider the function $f(x) = x^tQx$, namely

$$f(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

   a. What is the gradient of f?

   b. What is the Hessian matrix of f?

   c. What are the eigenvalues of Q?

d. What are the eigenvectors of Q?

e. Is f convex, concave, or neither?

f. Find a linear transformation $y_1=a_{11}x_1+a_{12}x_2$, $y_2=a_{21}x_1 + a_{22}x_2$ which allows the function to be written in separable form, $F(y) = y^tDy$, where D is a diagonal matrix, i.e.,

$$F(y) = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

g. Roughly sketch the graph of the equation F(y)= 10.

4. **Geometric Programming.** The stiffness of a wooden beam with rectangular cross-section is proportional to the product of its width (w) and the cube of its depth (d). We wish to find the dimensions of the stiffest beam which can be cut from a circular cylindrical log of diameter D.

a. Formulate a geometric programming model for the solution of this problem.

b. Formulate using only separable convex functions.

c. Write the Kuhn-Tucker conditions for the formulation in either (a) or (b).

d. Write the dual of the geometric programming problem (in either separable or nonseparable form).

e. Solve the dual problem.

f. What are the optimal values of w and d?

g. If the diameter D increases by 1%, estimate the increase in the optimal stiffness.

5. **Signomial Geometric Programming.** Consider the problem

$$\text{Minimize } \frac{x}{y} + \frac{yz}{x}$$

subject to

$$2\frac{y^2}{x} - z^2 \leq 1$$

$$x>0,\ y>0,\ z>0$$

a. Why is this not a posynomial geometric programming problem?

b. Show how the constraint may be approximated by a posynomial, by the use of "condensation". Perform the condensation at the point (x,y,z) = (1,1,1).

c. Is the feasible region of the condensed problem larger or smaller than that of the original problem?

d. What is the solution of this condensed problem?

e. Does this procedure, in which at each iteration the signomial is condensed at the optimal solution of the previous condensation, converge to an optimal solution of the original problem? Explain.