56:271 Nonlinear Programming
Final Examination -- December 16, 1988

Select any SIX of the EIGHT problems:

(1.) Unconstrained Search Methods: Consider the problem

Minimize f(x,y), where $f(x,y) = 2x^2 + y^2 - 8x - 2xy$

a. Test whether f(x,y) is convex, concave, both, or neither.
b. If we start with the initial "guess" $x^0 = 1$, $y^0 = 1$, and use the steepest descent method, in which direction will we first search?
c. The optimal stepsize in the direction of part (b) is $t^0 = 1/4$. What are the values of x and y at the start of iteration #2?
d. If we were to use the Fletcher-Reeves (i.e. "conjugate gradient") method, what would be the search direction in iteration #1? in iteration #2?
e. If we were to use Newton's method, what would be the search direction in iteration #1? What would be the stepsize used in this method?

(2.) Quadratic programming: Consider the problem

Minimize $1.5x^2 + xy + 0.5y^2 - 30x - 14y$
subject to $x + y \geq 3$
$2x - y \geq 4$
$0 \leq x$, $0 \leq y \leq 2$

a. Write the Kuhn-Tucker conditions for this problem, using explicit Lagrange multipliers for the non-negativity constraints.
b. Are the Kuhn-Tucker conditions [(i) necessary, (ii) sufficient, (iii) both, or neither ] for optimality?

The tableau for solution of this problem by Wolfe's method is:

Tableau (before adding artificial variable)

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | b |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 3 |
| 2 | ¯1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 4 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 2 |
| 3 | 1 | 1 | 2 | 0 | 0 | 0 | 0 | ¯1 | 0 | 30 |
| 1 | 1 | 1 | ¯1 | 1 | 0 | 0 | 0 | 0 | ¯1 | 14 |

c. How is this tableau related to the Kuhn-Tucker conditions you stated in part

## TABLEAU

(after pivoting slack and surplus variables into basis)

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | b |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 3 |
| 2 | ¯1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 4 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 2 |
| ¯3 | ¯1 | ¯1 | ¯2 | 0 | 0 | 0 | 0 | 1 | 0 | ¯30 |
| ¯1 | ¯1 | ¯1 | 1 | ¯1 | 0 | 0 | 0 | 0 | 1 | ¯14 |

**An artificial variable is now inserted:**

TABLEAU with artificial variable included

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | a | b |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 3 |
| 2 | ¯1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 4 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 |
| ¯3 | ¯1 | ¯1 | ¯2 | 0 | 0 | 0 | 0 | 1 | 0 | ¯1 | ¯30 |
| ¯1 | ¯1 | ¯1 | 1 | ¯1 | 0 | 0 | 0 | 0 | 1 | ¯1 | ¯14 |

**d. Explain how the coefficients of the artificial variable are determined.**

**The artificial variable is next entered into the basis:**

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | a | b |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 3 |
| 2 | ¯1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 4 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 |
| 3 | 1 | 1 | 2 | 0 | 0 | 0 | 0 | ¯1 | 0 | 1 | 30 |
| 2 | 0 | 0 | 3 | ¯1 | 0 | 0 | 0 | ¯1 | 1 | 0 | 16 |

**e. Explain how the pivot row was selected.**

**A pivot is next performed in row 2 & column 1:**

TABLEAU

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | a | b |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1.5 | 0 | 0 | 0 | 1 | ¯0.5 | 0 | 0 | 0 | 0 | 1 |
| 1 | ¯0.5 | 0 | 0 | 0 | 0 | 0.5 | 0 | 0 | 0 | 0 | 2 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 |
| 0 | 2.5 | 1 | 2 | 0 | 0 | ¯1.5 | 0 | ¯1 | 0 | 1 | 24 |
| 0 | 1 | 0 | 3 | ¯1 | 0 | ¯1 | 0 | ¯1 | 1 | 0 | 12 |

**f. Explain why (row 2, column 1) was selected for the pivot.**

**The next pivot is performed in row 5 & column 4:**

TABLEAU

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | a | b |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1.5 | 0 | 0 | 0 | 1 | ¯0.5 | 0 | 0 | 0 | 0 | 1 |
| 1 | ¯0.5 | 0 | 0 | 0 | 0 | 0.5 | 0 | 0 | 0 | 0 | 2 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 |
| 0 | 1.83333 | 1 | 0 | 0.666667 | 0 | ¯0.833333 | 0 | ¯0.333333 | ¯0.666667 | 1 | 16 |
| 0 | 0.333333 | 0 | 1 | ¯0.333333 | 0 | ¯0.333333 | 0 | ¯0.333333 | 0.333333 | 0 | 4 |

g. Explain why (row 5, column 4) was selected for the pivot.

After additional pivots, the tableau below was found:

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | a | b |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0.666667 | ¯0.333333 | 0 | 0 | 0 | 0 | 0.666667 |
| 1 | 0 | 0 | 0 | 0 | 0.333333 | 0.333333 | 0 | 0 | 0 | 0 | 2.33333 |
| 0 | 0 | 0 | 0 | 0 | ¯0.666667 | 0.333333 | 1 | 0 | 0 | 0 | 1.33333 |
| 0 | 0 | 1 | 0 | 0.666667 | ¯1.22222 | ¯0.222222 | 0 | ¯0.333333 | ¯0.666667 | 1 | 14.7778 |
| 0 | 0 | 0 | 1 | ¯0.333333 | ¯0.222222 | ¯0.222222 | 0 | ¯0.333333 | 0.333333 | 0 | 3.77778 |

h. What are the optimal values for the original variables? for the Lagrange multipliers which you defined in (a)?

**(3.) Lagrangian Duality:** Consider the problem

Minimize $x^2 + y^2$ subject to $x+y=1$

(note: no nonnegativity restrictions!)

a. Sketch a very rough graph and indicate the optimal solution $(x^*, y^*)$
b. Write the Kuhn-Tucker conditions for optimality.
c. Check whether the optimum indicated in (a) satisfies the Kuhn-Tucker cond
d. State the objective function of the Lagrangian dual problem (as a function o Lagrange multiplier l alone.) Is this function convex, concave, both, or ne
e. State the Lagrangian dual problem.
f. Solve the Lagrangian dual problem.
g. Is there a duality gap for this problem?

**(4.) Linearly-constrained minimization:** Consider the problem

Minimize $3x^2 + 2xy + y^2 - 30x - 14y$

subject to $x + y \geq 3$

$2x - y \geq 4$

$0 \leq x$, $0 \leq y \leq 2$

This problem was solved, using the GRG algorithm, which gave the following outpu

```
                    Generalized Reduced Gradient Algorithm
                    --------------------------------------

Please enter a feasible starting solution
(Be sure to include any slack/surplus variables you may have included!)
⎕:
     0 0 3 4
h(x) = 0 0
Please enter index set of 2 DEPENDENT variables
⎕:
     3 4
```

```
                              Iteration 1
                              -----------
x = 0 0 3 4
F(x)= 0
    Dependent Index Set: 3 4
    Independent Index Set: 1 2
h(x) = 0 0
Gradient = ¯30 ¯14 0 0
Negative of Reduced Gradient = 30 14
Search Direction = 30 14 ¯44 ¯46
(Normalized Search Direction = 0.652174 0.304348 ¯0.956522 ¯1)
Max Step Size = 3.13636
Optimal Step Size = 3.13636
   x = 2.04545 0.954545 0 0.863636
   h(x)=0 0
3 is replaced by 1 in Dependent Variable Set.
   h(x)=0 0
                              Iteration 2
                              -----------
x = 2.04545 0.954545 0 0.863636
F(x)= ¯57.3595
    Dependent Index Set: 1 4
    Independent Index Set: 3 2
h(x) = 0 0
Gradient = ¯15.8182 ¯8 0 0
Negative of Reduced Gradient = ¯15.8182 ¯7.81818
Search Direction = 7.81818 ¯7.81818 0 ¯23.4545
(Normalized Search Direction = 0.333333 ¯0.333333 0 ¯1)
Max Step Size = 0.863636
Optimal Step Size = 0.863636
   x = 2.33333 0.666667 0 0
   h(x)=0 0
4 is replaced by 2 in Dependent Variable Set.
   h(x)=0 0
                              Iteration 3
                              -----------
x = 2.33333 0.666667 0 0
F(x)= ¯59.4444
    Dependent Index Set: 1 2
    Independent Index Set: 3 4
h(x) = 0 0
Gradient = ¯14.6667 ¯8 0 0
Negative of Reduced Gradient = ¯10.2222 ¯2.22222

*** GRG HAS CONVERGED ***
                         |-------------------------------|
                         | Generalized Reduced Gradient  |
                         |     Solution                  |
                         |_______________________________|

x = 2.33333 0.666667 0 0
F(x) = ¯59.4444
∇F(x) = ¯14.6667 ¯8 0 0
h(x) = 0 0
```

**a. Why was the partition of variables into "dependent" and "independent" vari changed at the end of iteration #1?**

b. At iteration #2, show how, once M has been computed, the "reduced gradient" is computed.

c. Show how the search direction is computed in iteration #2, once the reduced gradient has been computed.

d. At iteration #3, why does the algorithm terminate?

**(5.) Gradient Projection Algorithm:** Consider again the problem of #4:

$$\text{Minimize } 3x^2 + 2xy + y^2 - 30x - 14y$$
$$\text{subject to } x + y \geq 3$$
$$2x - y \geq 4$$
$$0 \leq x, \quad 0 \leq y \leq 2$$

This problem was also solved using the Gradient Projection Algorithm, which gave output below. (Note that the upper bound on y is the third inequality constraint, w fourth & fifth inequalities of ≥ type are derived from the non-negativity constraints.)

```
                              Problem ID: Sample Problem
                              --------------------------
12/15/88  23:22
Please enter a feasible starting point for the search
⎕:
      0 0
X= 0 0
F(X) = 0
Constraint Partition: Tight: 4 5          Slack: 1 2 3
Matrix M =
                                        ¯1  0
                                         0 ¯1

Projection Matrix P =
 0  0
 0  0
Gradient ∇f(x) = ¯30 ¯14
Search Direction = 0 0
Lagrange Multipliers = ¯30 ¯14
***Release Tight Constraint 2
Constraint Partition: Tight: 4          Slack: 1 2 3 5
```

a. Explain how the matrix M is determined.

b. Explain the purpose of the projection matrix P, and its computation.

c. What is meant by "releasing" a tight constraint? Why was a tight constraint "released" in this case?

The output continues:

```
X= 0 0
F(X) = 0
Constraint Partition: Tight: 4          Slack: 1 2 3 5
Matrix M =
                                        ¯1  0

Projection Matrix P =
 0  0
 0  1
Gradient ∇f(x) = ¯30 ¯14
Search Direction = 0 14
Maximum step size = 0.142857
Optimal step size = 0.142857
```

d. Show how P is computed.
e. Explain how the search direction above is computed.
f. Under what conditions will this algorithm terminate?

**(6.) Dynamic programming with quadratic criterial & linear dynamics:** Consider the problem

$$\text{Minimize } (x_0^2 + 2y_0^2 + x_0) + (2x_1^2 - 0.5x_1y_1 + 2y_1^2 + x_1) + (2x_2^2 - 0.5x_2y_2 + y_2^2 + x_2 + y_2) + (x_3^2 + y_3^2 + x_3 + y_3) + x_4^2$$

subject to

$$x_{t+1} = 0.9x_t + y_t - 1, \quad t=0,1,2,3$$

$$x_0 = 10 \text{ (initial state)}$$

The problem was solved by dynamic programming, with the results below:

Cost Data

| i | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 2 | 1 | 0 | 0 |
| 1 | 2 | ¯0.5 | 2 | 1 | 0 | 0 |
| 2 | 2 | ¯0.5 | 1 | 1 | 1 | 0 |
| 3 | 1 | 0 | 1 | 1 | 1 | 0 |

where A[i] = coefficient of X[i]*2 D[i] = coefficient of X[i]
B[i] = coefficient of X[i]×Y[i] E[i] = coefficient of Y[i]
C[i] = coefficient of Y[i]*2 F[i] = constant

Cost of final stage: 1×X[N]*2 + 0×X[N] + 0

Transition data

| i | G | H | K |
|---|---|---|---|
| 0 | 0.9 | 1 | ¯1 |
| 1 | 0.9 | 1 | ¯1 |
| 2 | 0.9 | 1 | ¯1 |
| 3 | 0.9 | 1 | ¯1 |

Given the above data, the following vectors were computed:

Computed Arrays

Do you want the output routed to the printer?
n

| i | P | Q | R | S | T |
|---|---|---|---|---|---|
| 0 | 1.99425 | ¯1.82305 | 5.40452 | ¯0.55236 | 0.674407 |
| 1 | 3.17777 | ¯1.76507 | 3.64569 | ¯0.464765 | 0.449064 |
| 2 | 2.7101 | ¯0.932848 | 2.14501 | ¯0.42183 | 0.25 |
| 3 | 1.405 | ¯0.35 | 0.875 | ¯0.45 | 0 |
| 4 | 1 | 0 | 0 | 0.78418 | 1 |

Then, using $x_0 = 10$, the following optimum was computed:

```
|----------|
| Optimal  |
| Solution |
|__________|


    Final Exam '88
    --------------
i      Xi          Yi
- ---------- ----------
0 10          ¯4.73942
1  3.26058    ¯0.840998
2  1.09352    ¯0.0122165
3 ¯0.0280443   0.26262
4 ¯0.76262
```

X[i] = state variable, and Y[i] = decision variable, at stage i

a. Compute the optimal cost, based upon the computed arrays (instead of substi
the optimal solution into the original objective function!)

b. What is the terminal state ($x_4$) for the optimal solution?

c. If the initial state were 9, rather than 10, compute the optimal cost if possib
possible, explain how one would use the APL code to compute it.

Suppose that the optimal solution were incorrectly implemented, because the value $y_0 = -4$ were mistakenly used for the initial decision.

d. What value of the state variable $x_1$ would result?

e. What is now the best possible value for $y_1$, the decision variable at the next stage?

f. What is the minimum cost which can now be attained?

**(7.) Posynomial Geometric Programming:** Consider the problem

Minimize $1/(x^2 z y^2) + 2z$

subject to

$x^3 + y^2 + z \leq 1$

$x>0,\ y>0,\ z>0$

a. Write the dual of this geometric programming problem.

b. What is its "degree of difficulty"?

c. Write the separable, exponential form of the original primal problem. Are t
Kuhn-Tucker conditions necessary for this problem? Are they sufficient?

The results of the APL workspace GPLIB for this problem appears below. (Note:
algorithm used, the usual "weight" $\delta_j$ for term j in posynomial k is found by $\delta_j = w_j \lambda_k$)

## Primal Solution: Final Exam '88

Objective function: 21.1472

| i | X[i] |
|---|---|
| 1 | 0.465306 |
| 2 | 0.54774 |
| 3 | 0.599238 |

### Constraints

| k | P | Lambda |
|---|---|---|
| 2 | 1 | 3.08779 |

## Dual Solution: Final Exam '88

Weights of terms (ρ):

| k | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 9.4330E¯1 | 5.6704E¯2 | |
| 2 | 1.0074E¯1 | 3.0002E¯1 | 5.9924E¯1 |

Lagrange multipliers of primal constraints: 3.08779

Objective function:21.1384

Duality Gap: 0.00876489 = 0.0414471 %

**d. What are the optimal values of x,y, &z?**
**e. At the optimum, what fraction of the objective function is due to the first te**
**f. Given the dual variables appearing at the final iteration, together with the**
**objective value, write the equations which would be solved to find x, y, and z.**
**g. Suppose that the cost coefficient of the second term were to increase by 1%**
**to 2.02). Estimate the increase in the optimal cost. (Is this an under- or over-e**
**of the increase?)**

**(8.) Signomial Programming: Consider the problem**

**Minimize** $y^2z^4 - 5x^2$

**subject to**

$$5x^2/y^2 \leq 3z/y \quad - 2$$

$$x>0,\ y>0,\ z>0$$

**a. Formulate this as a standard signomial programming problem.**
**b. Write the dual of this problem. What is its "degree of difficulty"?**
**c. Solve the dual by an appropriate method.**
**d. Find the optimal primal solution.**