# 56:270 Linear Programming
## Midterm Examination - March 13, 1989

*Answer any four of the five problems:*

**(1.)** Below are several simplex tableaus. Assume that the objective in each case is to be **minimized**. Classify each tableau by writing to the right of the tableau a letter **A** through **G**, according to the descriptions below. Also answer the question accompanying each classification, if any.

(**A**) Nonoptimal, nondegenerate tableau. *Circle a pivot element which would improve the objective.*

(**B**) Nonoptimal, degenerate tableau. *Circle an appropriate pivot element. Would the objective improve with this pivot?*

(**C**) Unique optimum.

(**D**) Optimal tableau, with alternate optimum. *Circle a pivot element which would lead to another optimal basic solution.*

(**E**) Objective unbounded (below). *Specify a variable which, when going to infinity, will make the objective arbitrarily low.*

(**F**) Tableau with infeasible primal but feasible dual solution. *Circle an appropriate (dual simplex) pivot which would improve feasibility.*

(**G**) Tableau with both primal and dual solutions infeasible.

**Warning:** Some of these classifications might be used for several tableaus, while others might not be used at all!

| -z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | RHS |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 0 | -1 | 1 | 0 | 0 | 2 | 3 | -84 |
| 0 | 0 | 0 | -4 | 0 | 0 | 1 | 0 | 0 | 13 |
| 0 | 4 | 1 | 2 | -5 | 0 | 0 | 1 | 1 | 0 |
| 0 | -6 | 0 | 3 | -2 | 1 | 0 | 2 | 3 | 5 |

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| -z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | RHS |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 0 | 1 | -1 | 0 | 0 | -2 | 12 | -84 |
| 0 | 0 | 0 | -4 | 0 | 0 | 1 | 3 | 0 | 13 |
| 0 | 4 | 1 | 2 | -5 | 0 | 0 | 2 | 1 | 8 |
| 0 | -6 | 0 | 3 | -2 | 1 | 0 | -4 | 3 | 15 |

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| -z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | RHS |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 0 | 1 | -1 | 0 | 0 | 2 | 5 | -84 |
| 0 | 0 | 0 | -4 | 0 | 0 | 1 | 3 | 0 | -3 |
| 0 | 4 | 1 | 2 | -5 | 0 | 0 | 2 | 1 | 8 |
| 0 | -6 | 0 | 3 | -2 | 1 | 0 | -4 | 3 | 15 |

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| -z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | RHS |
|---|---|---|---|---|---|---|---|---|---|

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 0 | 1 | 3 | 0 | 0 | 2 | 0 | -84 | |
| 0 | 0 | 0 | -4 | 0 | 0 | 1 | 3 | 0 | 13 | ________ |
| 0 | 4 | 1 | -2 | -5 | 0 | 0 | 2 | 1 | -8 | |
| 0 | -6 | 0 | 3 | -2 | 1 | 0 | -4 | 3 | 15 | |

| -z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | RHS | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 0 | 1 | 1 | 0 | 0 | 2 | 0 | -84 | |
| 0 | 0 | 0 | -4 | 0 | 0 | 1 | 3 | 0 | 13 | ________ |
| 0 | 4 | 1 | 2 | -5 | 0 | 0 | 2 | 1 | 5 | |
| 0 | -6 | 0 | 3 | 2 | 1 | 0 | -4 | 3 | 8 | |

| -z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | RHS | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 0 | 1 | 4 | 0 | 0 | 2 | 2 | -84 | |
| 0 | 0 | 0 | -4 | 0 | 0 | 1 | -3 | 0 | 3 | ________ |
| 0 | 4 | 1 | 2 | -5 | 0 | 0 | 2 | 1 | 8 | |
| 0 | -6 | 0 | 3 | -2 | 1 | 0 | -4 | 3 | 15 | |

| -z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | RHS | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 0 | -1 | 3 | 0 | 0 | 0 | 0 | -84 | |
| 0 | 0 | 0 | -4 | 0 | 0 | 1 | 3 | 0 | 13 | ________ |
| 0 | 4 | 1 | 2 | -5 | 0 | 0 | 2 | 1 | 8 | |
| 0 | -6 | 0 | 3 | -2 | 1 | 0 | -4 | 3 | 15 | |

| -z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | RHS | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 0 | 1 | 3 | 0 | 0 | 2 | -2 | -84 | |
| 0 | 0 | 0 | -4 | 0 | 0 | 1 | 3 | 0 | 13 | ________ |
| 0 | -4 | 1 | 2 | -5 | 0 | 0 | -2 | 1 | 0 | |
| 0 | -6 | 0 | 3 | -2 | 1 | 0 | -4 | 3 | 5 | |

(2.) UPPER BOUNDING TECHNIQUE . Consider the following LP problem:

$$
\begin{array}{ll}
\text{MINIMIZE} & 3X_1 + 7X_2 + 4X_3 + 6X_4 \\
\text{subject to} & 3X_1 + X_2 \qquad\qquad - X_4 \geq 10 \\
& X_1 + 3X_2 + 2X_3 + 4X_4 \leq 10 \\
& 2X_1 - X_2 + 3X_3 + X_4 \geq 15 \\
& 1 \leq X_j \leq 5 \quad \text{for } j=1,2,3,4
\end{array}
$$

The APL output solving this problem using the Upper-Bounding Technique is attached. Please refer to it to answer the following questions.

(a.) What is the reduced cost of $x_2$ in the first iteration? Why was $x_2$ selected to enter the basis? Which other variables, if any, might have been selected to enter the basis and improve the objective?

(b.) As $x_2$ enters the basis in iteration 1, which variables increase? which decrease?

(c.) Explain the computation of the "blocking values" 5.000 and 13.333 in iteration 1.

(d.) Explain why the basis is unchanged in iteration #1.

(e.) Explain why, if the basis does not change, the basic solution in the second iteration differs from that at the first iteration. (Explain the computation of the basic variables.)

(f.) In iteration #2, explain how the "blocking values" 7.0 and 14.0 were computed.

(g.) In iteration #3, why was $X_7$ chosen to leave the basis?

(h.) If the product form of the inverse were being used, what "eta vector" would be generated in iteration #3? What is the corresponding pivot matrix?

(i.) Explain how the "substitution rates" (-3, -0.333, 0.667) were computed in iteration #4.

(j.) Explain how the simplex multipliers (0, 0, 1.333) were computed in iteration #4.

(k.) Explain why the algorithm may terminate with an "optimal solution" even though the reduced costs may not be all non-negative.

(l.) Using the upper bounding technique, what is the size of the basis inverse matrix? If we do not use the UBT, what would be the size of the basis inverse matrix?

(3.) LP USING LINEAR COMPLEMENTARY SOLUTION TECHNIQUE:
Consider the LP problem:

$$\begin{array}{ll} \text{MINIMIZE} & 5x_1 + 3x_2 \\ \text{subject to} & 5x_1 + 2x_2 \leq 36 \\ & x_1 - x_2 \geq 2 \\ & 3x_1 + x_2 \geq 12 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{array}$$

Consult the APL output on the following page to answer the following:

(a.) Write down the primal and the dual problems, both using only equality constraints.

(b.) Identify the constraints and label the columns with variable names in the initial tableau in the output.

(c.) State the complementary slackness conditions for this primal-dual pair of problems.

(c.) Suppose that we start with all slack and surplus variables in the basis. Is this solution feasible? Does it satisfy the complementary slackness conditions?

(d.) A single artificial variable is inserted into the tableau. Explain how the coefficients of this variable are selected.

(e.) Explain how it was decided where to pivot to enter the artificial variable into the basis. (circle the entry) What variable leaves the basis? Will complementary slackness be satisfied after this pivot?

(f.) How was it decided which variable was to be entered into the basis in the next iteration? Which variable leaves the basis? Is the complementary slackness condition satisfied after this pivot?

(g.) How is the next variable chosen to enter the basis?

(h.) How do you decide when to terminate? Is the solution which is given in the final tableau in the printed output <u>optimal</u>? If not, where would you pivot next?

(i.) What is the solution to the primal problem as given by the final tableau? the solution to the dual problem?

(4.) LINEAR PROGRAMMING DUALITY : Consider the following LP:

Minimize $2X_1 + 5X_2 + 3X_3 + X_5$

subject to $X_1 + 2X_3 - X_4 = 12$

$-X_1 + 2X_2 + X_4 + X_5$

$\leq 15$

$6X_2 - X_3 + 2X_5 \geq 8$

$X_1 \geq 0$, $X_2 \geq 0$, $1 \leq X_3 \leq 4$, $X_4 \leq 0$ ($X_5$ unrestricted in sign)

a. Write a dual of this LP problem.

b. The point (4,2,4,0,0) is feasible. What does this imply about the feasibility of the dual problem? about the boundedness of the dual problem?

c. Write the complementary slackness conditions for this primal-dual pair of problems.

(5.) Find the unknowns a through g in the current tableau (not necessarily optimal) below, given:

- $x_1$, $x_2$, and $x_3$ are slack variables
- the objective is: maximize $28x_4 + x_5 + 2x_6$

| basis \ | -z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | rhs |
|---|---|---|---|---|---|---|---|---|
| -z | 1 | b | c | 0 | 0 | -1 | g | -14 |
| $x_6$ | 0 | 3 | 0 | $-14/3$ | 0 | 1 | 1 | a |
| $x_2$ | 0 | 6 | d | 2 | 0 | 2.5 | 0 | 5 |
| $x_4$ | 0 | 0 | e | f | 1 | 0 | 0 | 0 |