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tststst 56:270 Linear Programming tststst
ststststs Final Examination ststststs
tststst May 11, 1998 tststst

- ***Write your name on the first page, and initial the other pages.***
- ***Answer all questions of Part One, and 3 (out of 4) problems from Part Two.***

		Possible	Score
Part One:	1. True/False & multiple choice	70	_____
	2. Sensitivity analysis (LINDO)	20	_____
	3. Upper bounding technique	20	_____
	4. Markov Decision Problem	15	_____
	Total possible:	125	_____

tststst PART ONE tststst

1. *Indicate whether true (+) or false (o). If false, <u>briefly</u> explain why, or give a counterexample.*

___ 1. If a primal LP has multiple optima, then the optimal dual solution must be degenerate.

___ 2. A pivot matrix is a product of elementary matrices.

___ 3. Given an LU factorization of the matrix A, the equation Ax=b (for any given vector b) may be solved by first solving Ly=b for vector y (backward substitution) and then Ux=y for vector x (forward substitution).

___ 4. Every echelon matrix must have an inverse.

___ 5. If E is an elementary matrix and A is a matrix of the same dimensions, then EA = AE.

___ 6. Gauss-Jordan elimination is a more efficient method for solving Ax=b (for a specific b) than is Gauss elimination with backsubstitution.

___ 7. Suppose that you perform Gaussian elimination of matrix A, performing k elementary row operations. If E_1, E_2, … E_k are the elementary matrices corresponding to these elementary row operations, then the matrix product $(E_1 E_2 \ldots E_k)A$ is an echelon matrix.

___ 8. If E is an elementary matrix and A is a matrix of the same dimensions, then the product EA yields the same result as performing the corresponding elementary row operation on the matrix A.

___ 9. The inverse of an elementary matrix is also an elementary matrix.

___ 10. If a primal minimization LP problem has a cost which is unbounded below, then the dual maximization problem has an objective which is unbounded above.

___ 11. If, when solving an LP by the simplex method, you make a mistake in choosing a pivot row, the resulting tableau is infeasible.

___ 12. If a (primal) simplex pivot does not result in an improvement of the objective function, then both basic solutions (before and after the pivot) must be degenerate.

___ 13. A "pivot" in a nonbasic column of a tableau will make it a basic column.

___ 14. A "pivot" in row i of the column for variable X_j will increase the number of basic variables.

___ 15. A "pivot" in the simplex method corresponds to a move from one corner point of the feasible region to another.

___ 16. Every basic solution of the problem "minimize cx subject to $Ax \leq b$, $x \geq 0$" corresponds to a corner of the feasible region.

___ 17. If, when solving an LP by the dual simplex method, you make a mistake in the minimum ratio test and choose the wrong ratio, the resulting pivot gives a (primal) infeasible solution.

___ 18. When you enter an LP formulation into LINDO, you must first convert all inequalities to equations.

___ 19. The restriction that X1 be nonnegative should be entered into LINDO as the constraint X1 >= 0.

___ 20. When you enter an LP formulation into LINDO, you must manipulate your constraints so that all variables appear on the left, and all constants on the right.

___ 21. If a primal LP has multiple optima, then its dual problem must also have multiple optima.

___ 22. If the optimal solution of a primal LP is degenerate, then the optimal dual solution must also be degenerate.

___ 23. If you increase the right-hand-side of a "less-than-or-equal" constraint in a minimization LP, the optimal objective value will either increase or stay the same.

___ 24. The "reduced cost" in LP provides an estimate of the change in the objective value when a right-hand-side of a constraint changes.

___ 25. In the two-phase simplex method, Phase One computes the optimal dual variables, followed by Phase Two in which the optimal primal variables are computed.

___ 26. At the completion of the revised simplex method applied to an LP, the simplex multipliers give the optimal solution to the dual of the LP.

___ 27. It is impossible for both the problems in a primal-dual pair to have unbounded objectives.

___ 28. It is impossible for both the problems in a primal-dual pair to be infeasible.

___ 29. The "minimum ratio test" is used to determine the pivot row in the (primal) simplex method.

___ 30. A variable that leaves the basis in some step of the primal simplex algorithm may be chosen as the variable to enter the basis at some later iteration.

___ 31. At the end of the first phase of the two-phase simplex method, the phase-one objective function must be zero if the LP is feasible.

___ 32. Consider a step of the primal simplex algorithm in which x_j is the entering variable. The minimum ratio in this step is the value of x_j in the basic solution that will be obtained by the pivot.

___ 33. In the dual simplex algorithm, the pivot element must be negative.

___ 34. .In the dual simplex algorithm, the (primal) variable to enter the basis is chosen before the variable to leave the basis.

___ 35. In the primal simplex algorithm, the variable to enter the basis is chosen before the variable to leave the basis.

___ 36. If a zero appears on the right-hand-side of row i of an LP tableau, then at the next simplex iteration you *cannot* pivot in row i.

___ 37. When maximizing in the simplex method, the value of the objective function cannot improve at the next pivot if the current tableau is degenerate.

___ 38. When minimizing in the simplex method, you must select the column which has the smallest (i.e., the most negative) reduced cost as the next pivot column.

___ 39. An LP with 8 variables and 3 equality constraints cannot have more than 50 basic feasible solutions.

___ 40. A "minimum ratio test" is used to determine the pivot row in the dual simplex method.

Multiple Choice: Write the appropriate letter (a, b, c, d, or e) : (*NOTA* =None of the above).

____ 41. If, in the optimal *primal* solution of an LP problem (min cx st Ax≤b, x≥0), there is zero slack in constraint #1, then in the optimal dual solution,

(a) dual variable #1 must be zero
(b) dual variable #1 must be positive
(c) slack variable for dual constraint #1 must be zero
(d) dual constraint #1 must be slack
(e) *NOTA*

____ 42. If, in the optimal *dual* solution of an LP problem (min cx st Ax≤b, x≥0), variable #2 is positive, then in the optimal primal solution,

(a) variable #2 must be zero
(b) variable #2 must be positive
(c) slack variable for constraint #2 must be zero
(d) constraint #2 must be slack
(e) *NOTA*

____ 43. If you make a mistake in choosing the pivot row in the simplex method, the solution in the next tableau

(a) will be nonbasic
(b) will be nonfeasible
(c) will have a worse objective value
(d) will be degenerate
(e) *NOTA*

____ 44. If you make a mistake in choosing the pivot column in the simplex method, the solution in the next tableau will

(a) be nonbasic
(b) be nonfeasible
(c) have a worse objective value
(d) be degenerate
(e) *NOTA*

____ 45. If there is a tie in the "minimum-ratio test" of the simplex method, the solution in the next tableau

(a) will be nonbasic
(b) will be nonfeasible
(c) will have a worse objective value
(d) will be degenerate
(e) *NOTA*

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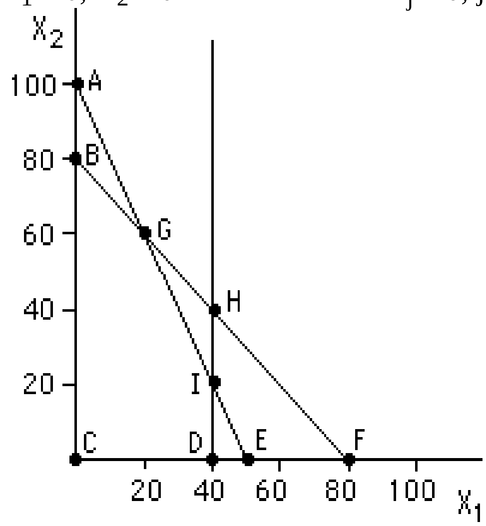
The problems (46)-(49) below refer to the following LP: *(with inequalities converted to equations:)*

Maximize $3X_1 + 2X_2$	Maximize $3X_1 + 2X_2$
subject to $2X_1 + X_2 \leq 100$	subject to $2X_1 + X_2 + X_3 = 100$
$X_1 + X_2 \leq 80$	$X_1 + X_2 + X_4 = 80$
$X_1 \leq 40$	$X_1 + X_5 = 40$
$X_1 \geq 0, X_2 \geq 0$	$X_j \geq 0$, j=1,2, 3,4,5



____ 46.The feasible region includes points
(a) I, D, & E (c) E, F, H, & I
(b) G, H, & I (d) B, D, G, I, & C (e) *NOTA*

____ 47. At point G, the basic variables <u>include</u> the variables
(a) X_2 & X_3 (c) X_1 & X_5
(b) X_3 & X_4 (d) X_1 & X_4 (e) *NOTA*

____ 48. Which point is degenerate in this problem?
(a) point B (c) point H
(b) point G (d) point I (e) *NOTA*

____ 49. <u>If</u> point G is optimal, then which dual variables must be zero, according to the Complementary Slackness Theorem?
(a) both Y_1 and Y_2 (d) Y_1 only
(b) both Y_1 and Y_3 (e) Y_2 only
(c) both Y_2 and Y_3 (f) Y_3 only

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DEA:

Note: *DMU* = "decision-making-unit"

___ 50. In the maximization problem of the primal-dual LP pair, the decision variables are:
a. The multipliers for the DMUs which are used to form a linear combination of DMUs in the reference set.
b. The "prices" assigned to the inputs and outputs.
c. The amount of each input and output to be used by the DMU
d. None of the above

___ 51. The "prices" or weights assigned to the input & output variables must
a. be nonnegative
b. sum to 1.0
c. Both a & b
d. Neither a nor b.

True (+) or false (o)?

___ 52. To perform a complete DEA analysis, an LP must be solved for *every* DMU.

___ 53. The optimal value of the LP cannot exceed 1.0.

___ 54. The efficiency scores of all DMUs are all computed based on a single set of "prices" of the inputs and outputs.

___ 55 . The number of inputs and outputs must be equal.

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Dantzig-Wolfe Decomposition:

___ 56. Each time a subproblem is solved,

a. its objective function may have changed.
b. its constraint coefficient matrix may have changed.
c. its feasible region may have changed.
d. none of the above
e. more than one of the above is true

True (+) or false (o)?

___ 57. The master problem's role is to determine "shadow prices" for the shared resources which will be used by subproblem proposals.

___ 58. The final solution, selected by the master problem, consists of one proposal from each of the subproblems.

___ 59. The variables in the master problem includes all of the variables which appear in the subproblems.

___ 60. The number of constraints in the master problem, plus the total number of constraints in the subproblems, is equal to the total number of constraints in the original formulatino of the problem. (Don't include simple nonnegativity constraints in these counts!)

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Separable Programming:

___ 61. In the "lambda" formulation of separable programming, there is one variable associated with each grid point.

___ 62. In the "delta" formulation of separable programming, there is one variable associated with each grid point.

___ 63. In both the "lambda" and "delta" formulations of separable programming, a "convexity" constraint is required for each piecewise-linear function, i.e. a constraint which requires the sum of the variables to be equal to one.

___ 64. In the "lambda" formulation of separable programming, the optimal solution cannot be less than the left-most nor greater than the right-most grid point.

___ 65. If the "restricted basis entry" rule is used in the "lambda" formulation of separable programming, at most one lambda will be basic for each piecewise-linear function.

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Stochastic Programming:

___ 66. In stochastic LP with recourse, the number of "scenarios" is equal to the number of random right-hand-sides times the number of possible values for each.

___ 67. Chance-constrained LP requires a constraint for every possible scenario.

___ 68. If Dantzig-Wolfe decomposition were applied to the dual of the two-stage stochastic LP, the number of subproblems would be equal to the number of random right-hand-sides.

___ 69. The value of the objective function of a chance-constrained LP is an *expected* value.

___ 70. If a right-hand-side of an LP constraint had a normal distribution, the number of scenarios required by the stochastic LP with recourse is infinite.

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2. Sensitivity analysis with LINDO:

Problem Statement: McNaughton Inc. produces two steak sauces, spicy Diablo and mild Red Baron. These sauces are both made by blending two ingredients A and B. A certain level of flexibility is permitted in the formulas for these products. Indeed, the restrictions are that:

i) Red Baron must contain no more than 75% of A.
ii) Diablo must contain no less than 25% of A and no less than 50% of B

Up to 40 gallons of A and 30 gallonsof B could be purchased. McNaughton can sell as much of these sauces as it produces at a price per gallon of $3.35 for Diablo and $2.85 for Red Baron. A and B cost

$1.60 and $2.05 per gallon, respectively. McNaughton wishes to maximize its net revenue from the sale of these sauces.

Define *D = gallons of Diablo to be produced*
R = gallons of Red Baron to be produced
A_1 *= gallons of A used to make Diablo*
A_2 *= gallons of A used to make Red Baron*
B_1 *= gallons of B used to make Diablo*
B_2 *= gallons of B used to make Red Baron*

The LINDO output for solving this problem follows:

```
 MAX     3.35 D + 2.85 R - 1.6 A1 - 1.6 A2 - 2.05 B1 - 2.05 B2
 SUBJECT TO
        2) - D + A1 + B1 =    0
        3) - R + A2 + B2 =    0
        4)   A1 + A2 <=   40
        5)   B1 + B2 <=   30
        6) - 0.25 D + A1 >=   0
        7) - 0.5 D  + B1 >=   0
        8) - 0.75 R + A2 <=   0
 END

          OBJECTIVE FUNCTION VALUE
  1)         99.0000000

VARIABLE            VALUE                      REDUCED COST
      D            50.000000                     0.000000
      R            20.000000                     0.000000
      A1           25.000000                     0.000000
      A2           15.000000                     0.000000
      B1           25.000000                     0.000000
      B2            5.000000                     0.000000

    ROW            SLACK OR SURPLUS          DUAL PRICES
      2)              0.000000                  -2.350000
      3)              0.000000                  -4.350000
      4)              0.000000                   0.750000
      5)              0.000000                   2.300001
      6)             12.500000                   0.000000
      7)              0.000000                  -1.999999
      8)              0.000000                   2.000000

RANGES IN WHICH THE BASIS IS UNCHANGED

                                          OBJ COEFFICIENT RANGES
    VARIABLE         CURRENT          ALLOWABLE           ALLOWABLE
                        COEF           INCREASE            DECREASE
        D             3.350000        0.750000            0.500000
        R             2.850000        0.500000            0.375000
        A1           -1.600000        1.500001            0.666666
        A2           -1.600000        0.666666            0.500000
        B1           -2.050000        1.500001            1.000000
        B2           -2.050000        1.000000            1.500001

                                            RIGHTHAND SIDE RANGES
      ROW            CURRENT            ALLOWABLE         ALLOWABLE
                         RHS              INCREASE           DECREASE
```

2	0.000000	10.000000	10.000000
3	0.000000	16.666668	3.333333
4	40.000000	50.000000	10.000000
5	30.000000	10.000000	16.666664
6	0.000000	12.500000	INFINITY
7	0.000000	6.250000	5.000000
8	0.000000	2.500000	12.500000

THE TABLEAU:

ROW	(BASIS)	D	R	A1	A2
1	ART	0.000	0.000	0.000	0.000
2	A1	0.000	0.000	1.000	0.000
3	R	0.000	1.000	0.000	0.000
4	A2	0.000	0.000	0.000	1.000
5	B2	0.000	0.000	0.000	0.000
6	SLK 6	0.000	0.000	0.000	0.000
7	D	1.000	0.000	0.000	0.000
8	B1	0.000	0.000	0.000	0.000

ROW	B1	B2	SLK 4	SLK 5	SLK 6
1	0.000	0.000	0.750	2.300	0.000
2	0.000	0.000	-0.500	1.500	0.000
3	0.000	0.000	2.000	-2.000	0.000
4	0.000	0.000	1.500	-1.500	0.000
5	0.000	1.000	0.500	-0.500	0.000
6	0.000	0.000	-0.250	0.750	1.000
7	0.000	0.000	-1.000	3.000	0.000
8	1.000	0.000	-0.500	1.500	0.000

ROW	SLK 7	SLK 8	RHS
1	2.000	2.000	99.000
2	3.000	2.000	25.000
3	-4.000	-4.000	20.000
4	-3.000	-2.000	15.000
5	-1.000	-2.000	5.000
6	2.000	1.000	12.500
7	4.000	4.000	50.000
8	1.000	2.000	25.000

a. How many gallons of Diablo are produced? _______________

b. How much profit does the firm make on these two products? _____________

c. What *additional* amount should the firm be willing to pay to have another gallon of ingredient B avai
____

d. What is the *total* amount the firm should be willing to pay for another gallon of ingredient B? ____

e. How many gallons should they be willing to buy at this cost? _______________

f. How much can the price of Diablo increase before the composition of the current optimal product mi
changes? _________________

g. If one *fewer* gallon of ingredient B were available to be used, what would be
the quantity of Diablo produced? ___________gallons *circle:* (increase / decrease)
the quantity of Red Baron produced? __________ gallons *circle:* (increase / decrease)

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3. UPPER BOUNDING TECHNIQUE. Consider the following LP problem:

$$\text{MINIMIZE} \quad 2X_1 + X_2 + X_3 - X_4$$

$$\text{subject to} \quad \begin{aligned} 2X_1 - X_2 + X_3 - 2X_4 &\le 18 \\ -X_1 + X_2 - 2X_3 + 3X_4 &\le 18 \\ 3X_1 + X_2 - X_3 &\ge 18 \\ 0 \le X_1 &\le 5 \\ 2 \le X_2 &\le 4 \\ 0 \le X_3 &\le 9 \\ 1 \le X_4 &\le 2 \end{aligned}$$

The APL output solving this problem using the **Upper-Bounding Technique** is attached. The starting solution (found perhaps by a Phase One procedure) is $x_1=5$, $x_2=4$, $x_3=1$, $x_4=2$. Please refer to the APL output to answer the following questions.

Constraints

1	2	3	4	5	6	7	b
2	¯1	1	¯2	1	0	0	18
¯1	1	¯2	3	0	1	0	18
3	1	¯1	0	0	0	1	18

Objective & Bounds

*	1	2	3	4	5	6	7
c[i]	2	1	1	¯1	0	0	0
L[i]	0	1	0	2	0	0	0
U[i]	3	4	10	5	100	100	100

Optimization type: Minimization

Iteration 1

```
Current partition:
B= 3 5 6 /  L= 7 /  U= 1 2 4
Basis inverse matrix =
                       0 0 ¯1
                       1 0  1
                       0 1 ¯2
Basic solution= 5 4 1 2 15 15 0 with Z = 13
Simplex multipliers= 0 0 ¯1
Reduced costs= [    ] 2 0 ¯1 0 0 1

Entering variable is X[1] from set U
Substitution Rates= ¯3 5 ¯7
--------------------
Decreasing variables:     3     [    ]
Block at value:         .333   2.143
--------------------
Block at X[3] at value 0.3333
```

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Iteration 2

```
Current partition:
B= 1 5 6 ⁄  L= 7 3 ⁄  U= 2 4
Basis inverse matrix =
                        0 0  0.3333
                        1 0 ¯0.6667
                        0 1  0.3333

Basic solution= [      ] 4 0 2 16.6667 12.6667 0 with Z = 11.3333
Simplex multipliers= 0 0 0.6666666667
Reduced costs=    0 0.3333 1.6667 ¯1 0 0 ¯0.6667

Entering variable is X[7] from set L
Substitution Rates= 0.3333       ¯0.6667 0.3333
--------------------
Decreasing variables:       1       1
Block at value:        14.000  38.000
--------------------
Block at X[1] at value 14
```

Iteration 3

```
Current partition:
B= 7 5 6 ⁄  L= 3 1 ⁄  U= 2 4

Basis inverse matrix =
                       0 0 1
                       1 0 0
                       0 1 0
Basic solution= 0 4 0 2 26 8 14 with Z = 2
Simplex multipliers= 0 0 0
Reduced costs=    2 1 1 ¯1 0 0 0

Entering variable is X[2] from set U
Substitution Rates= 1 [    ] [    ]
--------------------
Decreasing variables:       1
Block at value:        26.000
--------------------
Variable does NOT enter basis
```

. .

```
                              Iteration 4
Current partition:
B= 7 5 6 /  L= 3 1 2 /  U= 4
Basis inverse matrix =
                         0 0 1
                         1 0 0
                         0 1 0
Basic solution= 0 2 0 2 24 10 16 with Z = 0

Simplex multipliers= 0 0 0
Reduced costs=    2 1 1 ¯1 0 0 0
                              Optimal Solution
                             ------------------
i         1       2       3       4       5       6       7
X[i]    .000   2.000    .000   2.000  24.000  10.000  16.000

Objective Z= 0
```

(a.) What is the reduced cost of x_1 in the first iteration? _____
Why was x_1 selected to enter the basis?

(b.) Which other variables, if any, might have been selected instead of x_1 to enter the basis in the first iteration in order to improve the objective? _________________

(c.) As x_1 enters the basis in iteration 1,
which variables increase? ____________
which variables decrease? ____________

(d.) Explain how the "blocking values" 0.333 and 2.143 were computed in iteration #1.

(e.) Explain why x_3 was selected to leave the basis in iteration #1.

(f.) At the beginning of iteration #2, what is the value of x_1?

(g.) As x_2 enters the basis in iteration #3, is it increasing or decreasing? ___________

(h.) What are the substitution rates of x_2 in iteration #3?
_1__ for variable x__,
____ for variable x__, and
____ for variable x__.

(i.) As x_2 is changed in iteration #3, which variables increase? _________
Which variables decrease? ___________

(j.) In iteration #3, why does x_2 **not** enter the basis?

(k.) Explain why, if the basis does not change in iteration #3, the basic solution at the fourth iteration differs from that at the third iteration.

(l.) Explain why the algorithm terminates in iteration #4 with an "optimal solution" even though the reduced costs are not all non-negative.

4. Markov Decision Problem: A component of a machine is inspected at the end of each day, and classified into one of three states:

1) GOOD Condition
2) DETERIORATED, but usable
3) BROKEN, machine is inoperable

If in GOOD condition (state 1), it will again be in **state 1** (good) condition at the end of the following day with probability 0.7, in **state 2** with probability 0.2, and in **state 3** with probability 0.1.

If in DETERIORATED condition (state 2), and no repair is done, it will again be in **state 2** with probability 0.6, otherwise it will be in **state 3**. If it is repaired, however, it will begin the next day in GOOD condition (state 1) but may, of course, deteriorate during the day.

If in BROKEN condition (state 3), it may be either repaired or replaced. If replaced, it will be in state 1 at the beginning of the next day. If repaired, it begins the next day in state 2.

Cost of a repair is $100 if in state 2 and $200 if in state 3.

Cost of a replacement is $700.

A machine produces a profit of $200 if it operates all day. Assume, however, that a machine which breaks and is in condition 3 at the end of the day has earned no profit during that day.

Consider now the Markov Decision Problem LP model to find the optimal policy, i.e., that which maximizes the average daily profit. Some APL output for this model is show below. (Note, however, that the MDP model is *minimizing the negative of the profit*, because the APL code assumes minimization!)

States:

i	name
1	Good
2	Deteriorated
3	Broken

Actions:

k	name
1	Do nothing
2	Repair
3	Replace

Cost Matrix

		States		
k	name	1	2	3
1	Do nothing	¯200	¯200	9999
2	Repair	9999	¯100	0
3	Replace	9999	9999	700

(Rows ~ actions, Columns ~ states)
A value of 9999 above signals an infeasible action in a state.

Transition Probabilities

Action: Do nothing

from \ to	1	2	3
1	0.7	0.2	0.1
2	0	0.6	0.4
3	0	0	1

Action: Repair

from \ to	1	2	3
1	1	0	0
2	0.7	0.2	0.1
3	0	0.6	0.4

Action: Replace

from \ to	1	2	3
1	1	0	0
2	0	1	0
3	0.7	0.2	0.1

e. What is the meaning of the LP decision variable X_2^3 in the MDP model?

f. Several values have been omitted from the LP tableau for the model. Insert these missing values.

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LP Tableau for MDP

							RHS
k~action →	k:	1	1	2	2	3	
i~state →	i:	1	2	2	3	3	
	Min	¯200	[ ]	¯100	0	700	0
		0.3	0	¯0.7	0	¯0.7	0
		¯0.2	0.4	[ ]	¯0.6	¯0.2	[ ]
		1	1	1	1	1	[ ]

g. How many basic variables will this LP have, in addition to the objective value (-z)? _____

h. List (the names of) the basic variables of this LP which correspond to the policy of replacing a part in state 3, but otherwise doing nothing. ___________________________________

Suppose that we begin with the initial basic feasible LP solution corresponding to the policy: “Do nothing unless broken, in which case repair”. The initial tableau is shown below.

i. Following this policy, what is the average profit per day? ___________

j. Following this policy, what is the fraction of the days in which the component will be found to be broken when inspected? ___________

Iteration 0

k:	1	1	2	2	3	
i:	1	2	2	3	3	rhs
Min	0	0	¯100	0	500	120
	1	0	¯2.333333333	0	¯2.333333333	0
	0	1	2.333333333	0	1.333333333	0.6
	0	0	1	1	2	0.4

k. Based upon the above LP tableau, how should this policy be changed in order to improve the average daily profit? That is, what is a policy which should increase the average profit per day?
If condition is "good", __________________
If condition is "deteriorated", __________________
If condition is "broken", ___________________

l. Which variable will enter the basis? _________
Which variable will leave the basis? _________