°@°@° 56:270 Linear Programming °@°@°
Final Exam - May 4, 1989
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Select any 7 of the 9 problems below:

(1.) ANALYSIS OF MPSX OUTPUT: Please refer to the attached materials on the MULTISTAGE PRODUCTION PLANNING (problem statement, formulation, and MPSX output). Answer the following questions. (If there is insufficient information in the MPSX output, simply answer NOT ENOUGH INFORMATION):

(a.) If the June demand for fiberglass ("glass") tires were to increase by 10%,
- i) what is the effect on the objective function?
- ii) does the basis change?
- iii) what changes result in the number of tires produced on the Wheeling & Regal machines, i.e., the production variables $W_{n,1}$, $W_{g,1}$, $R_{n,1}$ , and $R_{g,1}$?

(b.) In the current solution, 53.3333 hours on the Regal machine are scheduled in August. What is the effect on the cost of rounding this down to 53 hours? up to 54 hours? Which is preferable?

(c.) Inventory storage costs for Nylon tires at the end of June are $0.10 per tire. How much may this cost increase before it is necessary to reduce the number of nylon tires stored? If this happened, how many nylon tires would be stored? Would this change the number of nylon tires produced in July on the Wheeling machine? on the Regal machine?

(2.) DANTZIG-WOLFE DECOMPOSITION . We wish to use the decomposition technique to solve the following problem:

MINIMIZE $-2X_1 - 3X_2 - 4X_3 - 3X_4$

subject to

$$
\begin{aligned}
X_1 + 2X_2 + 2X_3 + X_4 &\leq 50\\
-X_1 + X_2 + X_3 + X_4 &\leq 30\\
X_1 + 3X_2 &\leq 45\\
2X_1 + X_2 &\leq 35\\
X_3 &\leq 20\\
X_4 &\leq 20\\
2X_3 - X_4 &\leq 20\\
X_3 + X_4 &\leq 25\\
X_1, X_2, X_3, \text{ and } X_4 &\geq 0
\end{aligned}
$$

It was decided to use two subproblems, one with variables $(X_1,X_2)$ and the other with variables $(X_3,X_4)$, writing the feasible region of each as a combination of its extreme points. (That is, there will be two sets of λ's and two "convexity" constraints.) The APL output solving this problem is attached. Since $X_1=X_2=X_3=X_4=0$ is feasible in this problem, we initially add $X_1=X_2=0$ to the master problem as proposal #1, and $X_3=X_4=0$ as proposal #2.

(a.) Write the first master problem tableau (after adding proposals 1 and 2).
(b.) Are proposals 3 and 4 simultaneously feasible, i.e., is it possible to implement both of these proposals? Why or why not?
(c.) Write the second master problem tableau (after adding proposals 3 and 4).
(d.) If we were to stop before solving the second master problem tableau, how close are we guaranteed to be to the optimum?
(e.) What are the values of $X_1$ through $X_4$ found by the second master problem? (Note: these values were blanked out in the output!)
(f.) What is the objective function for subproblem 2 at the next iteration (i.e. after the second master problem has been solved)?
(g.) Why does the algorithm terminate where shown? What is the optimal solution of the original problem?
(h.) In the optimal solution, is any subproblem proposal used "as is", i.e. without modification?

(3.) STOCHASTIC PROGRAMMING. The Concrete Products Corporation has the capability of producing 4 types of concrete blocks. Three processes are required, along with cement, with the following usages per pallet:

| Block type: | 1 | 2 | 3 | 4 | Resource avail. |
|---|---|---|---|---|---|
| Batch mixing (hrs) | 1 | 2 | 10 | 16 | 84 hrs/month |
| Mold vibrating (hrs) | 1.5 | 2 | 4 | 5 | 150 hrs/month |
| Inspection | .5 | .6 | 1 | 2 | 48 man-hrs/mo. |
| Cement required (bags) | 2 | 3 | 5 | 6 | 120 bags. |
| Profit ($/pallet) | 8 | 14 | 30 | 50 | |

The company wishes to plan the number of pallets of each type block to be produced during the next month.

(a.) CHANCE CONSTRAINED PROGRAMMING: Suppose that the available hours given above are the expected values for normal distributions, with standard deviations of 5, 2, and 4, respectively.
(i.) Reformulate the problem with the restriction that enough hours are available for each process with at least 95% probability. (The Standard Normal CDF evaluated at 1.65 is approximately 0.95)
(ii.) If the resulting production plan is implemented, what is the probability that it is infeasible because of a shortage of hours in one or more processes?

(b.) STOCHASTIC LP WITH RECOURSE: Now suppose that bids have been made for two contracts, which, if the winning bids, would reduce the hours available in the three processes for the production of the 4 block types above. Bid #1, if successful, would require 10, 10, and 5 hours, respectively, in mixing vibrating, and inspection processes. Bid #2, if successful, would require 5 hours of each process. The company estimates that it has a 50% probability of winning the first contract, but only 25% probability of winning the second.

The monthly production of the four block types must be planned before learning whether the bids are successful. After learning this, you have several recourses available to you:

- Reduce scheduled production of block #1 or 2 (at a cost of 50¢/pallet in addition to losing the specified profit)
- Increase scheduled production of block #1 or 2 (but with a reduction of $1/pallet in profit.)
- Purchase additional bags of cement, at $1/bag

Formulate an LP model to compute the production plan which maximizes the expected profit, as well as the recourse to be used in the event of each possible outcome.

(4.) LP USING COMPLEMENTARY PIVOTING TECHNIQUE. Consider the original LP model given in problem (3) for the Concrete Products Corp. problem. Attached to your exam is the APL output.

(a.) Write down the primal and the dual problems, both using only equality constraints.
(b.) Write down the tableau containing both sets of equality constraints from part (a.)
(c.) We start with slack and surplus variables in the basis (negating rows with surplus variables as required). Is this solution primal feasible? dual feasible? Does it satisfy complementary slackness?

A single artificial variable was defined, and its column inserted into the tableau (tableau #3).

(d.) Explain how the next pivot element (in tableau #3) was selected. What variable leaves the basis? Will complementary slackness be satisfied after this pivot?
(e.) What variable or variables (in tableau #4) may now enter the basis?
(f.) How was the pivot location in tableau #5 selected?
(g.) How do you decide when an optimal solution has been found, since you have no row of reduced costs?
(h.) If the primal problem is unbounded, how does the algorithm terminate?
(i.) If the primal problem is infeasible, how does the algorithm terminate?
(j.) What is the optimal solution for the Concrete Products Corporation?

(5.) LP FORMULATION: Consider a three-period production planning problem in which there can be both regular-time and overtime production in each period.

- In every period, the regular-time capacity is 2 units and the overtime capacity is 2 units.
- A raw material is used in the production process, one unit of raw material per unit of product.
- Five units of the raw material are received each period; any unused material may be stored for future use, but a storage cost of $0.50/unit is incurred.
- Demands in periods 1, 2, and 3 are 2, 7, and 3 units, respectively.
- Regular-time production costs $4/unit, while overtime production costs $7/unit.
- Inventory holding costs are $1 per unit per period. There is 1 unit in inventory initially, and the company wishes to have 1 unit in inventory at the end of the third period.
- No shortages are allowed.

(a.) Formulate an LP model for this problem. What are the total numbers of variables & constraints required?
(b.) Suppose that shortages may be backordered, i.e. demand may be satisfied by shipping goods to the customer later than the period demanded. Doing so incurs a penalty of $2/unit per period late, however. Modify your LP model to allow backorders.

(6.) PARAMETRIC PROGRAMMING: Consider again the original Concrete Products Corporation problem in (3.) Suppose that there is a threat of a strike during the month. Each shift that the factory is on strike reduces the available hours in mixing, vibrating, and inspection processes by 4 hrs., 6 hrs., and 2 man-hrs., respectively. Parametric programming was done on the right-hand-side, with the number of shifts eliminated as the parameter, $\lambda$, so as to plan for any eventuality. Refer to the attached APL output to answer the following questions:

(a.) Consider the original optimal tableau (tableau #2), with the parameter $\lambda = 0$. Write the basic solution as a function of $\lambda$. For what values of $\lambda$ is this basis feasible?

(b.) If $\lambda$ is increased from 0 to its upper bound found in (a), which variable should enter the basis? Which variable will it replace? Explain why.

(c.) What is the range of values of $\lambda$ for which this new basis (i.e., tableau #3) is feasible? If $\lambda$ increases to the upper limit of this range, what variable enters the basis? What variable leaves the basis? Explain why.

(d.) What is the range of values of $\lambda$ for which this new basis (i.e., tableau #4) is feasible? If $\lambda$ increases to the upper limit of this range, where should you pivot? Explain.

(e.) Draw a rough sketch of the variables $x_1$ through $x_4$ as a function of $\lambda$, the number of production shifts on strike.

(7.) The following is the current simplex tableau of an LP problem. The objective is to minimize $-2x_1 - 3x_2 - 3x_3$, and $x_4$, $x_5$, and $x_6$ are slack variables.

| -z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | rhs |
|---|---|---|---|---|---|---|---|
| 1 | a | 0 | b | 1 | c | 0 | 70 |
| d | 1 | 0 | -1.6 | e | -.4 | 0 | 8 |
| 0 | 0 | 1 | 2.4 | .2 | .6 | 0 | f |
| 0 | 0 | g | 13 | 0 | 2 | h | 70 |

a. Find the values of the unknowns a through h in the tableau, where possible.
b. What is the inverse of the current basis matrix?
c. What are the values of the current simplex multipliers?
d. What are the values of the basic variables?
e. Is the current tableau optimal?

## (8.) LINEAR PROGRAMMING DUALITY :

(a.) Consider the following LP:

$$\begin{array}{ll} \text{Maximize} & x_1 - 2x_2 + x_3 \\ \text{s.t.} & 2x_1 + 7x_2 - x_3 \leq 3 \\ & x_1 + x_2 + x_3 \geq 1 \\ & x_1 \geq 0,\ x_2 \text{ unconstrained in sign},\ x_3 \leq 0 \end{array}$$

Write a dual problem for this LP.

(b.) Sketch the feasible region for the dual problem in (a) and solve graphically.

(c.) Write the complementary slackness conditions for the primal/dual pair above.

(d.) What is the solution of the primal problem in (a)?

(e.) For each of the following 4 situations, give an example primal/dual pair of LPs, if possible (primal &/or dual need have only one variable):

| | Primal (maximization) | Dual (minimization) |
|---|---|---|
| ________ | inconsistent | unbounded |
| | unbounded | inconsistent |
| | unbounded | unbounded |
| | inconsistent | inconsistent |

## (9.) MISCELLANEOUS:

(a.) What is an LP "basis"? What is the basis of a vector space? What is the relationship between these concepts?

(b.) What is a "basic variable"? a "basic solution"?

(c.) Is the LP optimal solution always basic?

(d.) Does a basic solution always have exactly m positive variables (where m = # of constraints, excluding nonnegativity restrictions)?

(e.) In the upper bounding technique, are all nonbasic variables equal to zero? Explain.

(f.) Suppose that a variable is unrestricted in sign. Explain how the revised simplex method can handle this (without replacing the variable with a pair of nonnegative variables).

(g.) A free variable may be replaced by substituting the difference of two nonnegative variables, e.g. $x_i = x_i^+ - x_i^-$, where $x_i^+ \geq 0$ and $x_i^- \geq 0$. Explain why, at every iteration, both $x_i^+$ and $x_i^-$ cannot be basic simultaneously.

(h.) Discuss conditions under which you would choose to use the dual simplex method rather than the primal simplex method.

(i.) If the optimal primal solution is degenerate, what can be said about the optimal dual solution, if anything?