56:270 LINEAR PROGRAMMING

FINAL EXAMINATION - MAY 12, 1986

SELECT THREE PROBLEMS (OF A POSSIBLE FOUR) FROM PART ONE, AND THREE PROBLEMS (OF A POSSIBLE FOUR) FROM PART TWO.

PART ONE: SCORE:	1	2	3	4	TOTAL	GRAND TOTAL:
PART TWO: SCORE:	1	2	3	4	TOTAL	________

PART ONE

1. REVISED SIMPLEX METHOD: Consider the LP problem:

$$\begin{array}{ll} \text{Minimize} & 2X_1 + 5X_2 + 7X_4 + 15X_5 + 14X_6 \\ \text{subject to} & X_1 + 2X_2 - X_3 + X_4 + 4X_5 + 5X_6 = 10 \\ & X_1 + 3X_2 - 2X_3 + 2X_4 + 5X_5 + 7X_6 = 12 \\ & X_j \geq 0,\ j=1,2,\ldots 6 \end{array}$$

The current basis is B = {3,5}.

(a.) What is the current basis matrix?

(b.) What is the basis inverse matrix?

(c.) What is the current basic solution?

(d.) What are the values of the simplex multipliers?

(e.) Price the second column of the coefficient matrix. Would entering this column into the basis matrix result in an improvement in the solution?

(f.) Assume that column 2 is to be entered into the basis (regardless of whether doing so improves the solution). What is the "updated" column, i.e. the column of substitution rates?

(g.) Which column of the current basis will be replaced by column 2?

(h.) If the product form of the inverse is being used, what pivot matrix will now be added to the list? What is the corresponding "ETA" vector?

(i.) If the variables X_1 through X_6 above all have upper bounds of 10, what is now your answer to part (g)?

(j.) How many basic (including infeasible) solutions exist for the above problem?

(k.) If the objective were to be maximized rather than minimized, how would your answers to parts (e), (f), and (g) change, if at all?

(2.) Write down the dual of the following LP problem:

$$
\begin{array}{ll}
\text{MAXIMIZE} & 5X_2 + 10X_3 + X_4 - 8X_5 \\
\text{subject to:} & -X_1 - 13X_2 + 45X_3 + 16X_5 - 7X_6 \le 89 \\
 & 3X_3 - 18X_4 + 30X_7 \ge 37 \\
 & 4X_1 - 5X_3 + X_6 = -13 \\
 & 2 \le X_1 \le 10 \\
 & 16 \le X_3 \\
 & X_5 \le 0 \\
 & X_6 \text{ unrestricted in sign} \\
 & X_2, X_4, \text{ and } X_7 \ge 0
\end{array}
$$

(3.) DUAL SIMPLEX METHOD: Consider the simplex tableau below:

-Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	RHS	
1	2	4	3	1	5	0	0	0	0	(MIN)
0	1	-2	-1	1	1	1	0	0	3	
0	-1	-1	-1	1	1	0	1	0	-4	
0	1	1	-2	2	-3	0	0	1	-2	

(a.) What is the current primal solution? Is it feasible? Does it satisfy the primal optimality conditions?

(b.) Circle every element in the tableau which the dual simplex method might select as a pivot element.

(4.) UPPER BOUNDING TECHNIQUE. Consider the following LP problem:

$$
\begin{array}{ll}
\text{MINIMIZE} & 2X_1 + X_2 + X_3 - X_4 \\
\text{subject to} & 2X_1 - X_2 + X_3 - 2X_4 \leq 6 \\
& -X_1 + X_2 - 2X_3 + 3X_4 \leq 9 \\
& 3X_1 + X_2 - X_3 \geq 3 \\
& 0 \leq X_1 \leq 5 \\
& 2 \leq X_2 \leq 4 \\
& 0 \leq X_3 \leq 9 \\
& 1 \leq X_4 \leq 2
\end{array}
$$

The APL output solving this problem using the Upper-Bounding Technique is attached. Please refer to it to answer the following questions.

(a.) Explain why in the initial basic solution, i.e.

X_1	X_2	X_3	X_4	X_5	X_6	X_7
5	2	0	1	0	9	14

the value of the basic variables (X_5, X_6, X_7) are NOT given by the quantity $(A^B)^{-1}b$, which is equal to (6, 9, -3).

(b.) In Iteration #2, explain how it is determined that, as the nonbasic variable X_4 is increased, the basic variable X_5 increases, while the basic variable X_6 decreases.

(c.) In this iteration, explain how the "blocking values" 45.333 and 1.444 were computed.

(d.) Why does X_4 not enter the basis?

(e.) Explain why, if the basis does not change, the basic solution at the third iteration differs from that at the second iteration.

(f.) Explain why the algorithm terminates with an "optimal solution" if the reduced costs are not all non-negative.

PART TWO

(1.) SEPARABLE PROGRAMMING: Consider the nonlinear programming problem

MINIMIZE $f_1(X_1) + f_2(X_2)$

subject to

$$X_1 + X_2 \geq 75$$
$$2X_1 - X_2 \geq 100$$
$$X_1 + 2X_2 \geq 300$$
$$0 \leq X_1 \leq 200, \quad 0 \leq X_2 \leq 200$$

where the functions f_1 and f_2 are piecewise linear as shown below.

The "lambda" formulation of the problem was first defined. The initial tableau is shown below.

(a.) Express X_1 and X_2 in terms of the λ's

(b) Express $f_1(X_1)$ and $f_2(X_2)$ in terms of the λ's

(c.) When the ordinary simplex method is used, the solution found is $\lambda_{11} = 0.4286$, $\lambda_{15} = 0.5714$, $\lambda_{21} = \lambda_{25} = 0.5$ (other variables being zero). What is wrong with this solution?

(d.) Below are reproduced two tableaux, obtained by using a "restricted basis entry" rule. What is the corresponding value of X_1 and of X_2 for each tableau?

(e.) Indicate (by circling) every possible pivot element in this tableau which might improve the solution (if any).

(f.) Reformulate the problem, using the "delta" formulation method. Write X_1 and X_2 as expressions in the delta variables.

(g.) What are the values of the "delta" variables corresponding to the second solution in the "lambda" tableau of part (e)?

(h.) If we are using the "upper bounding technique", which delta variables are in the basis at this iteration? Which delta variables are candidates for entry into the basis (i.e. variables for which the reduced cost must be computed)?

(2.) DANTZIG-WOLFE DECOMPOSITION. We wish to use the decomposition technique to solve the following problem:

$$
\begin{array}{llr}
\text{Maximize} & 15X_1 + 7X_2 + 15X_3 + 20Y_1 + 12Y_2 & \\
\text{subject to} & X_1 + X_2 + X_3 + Y_1 + Y_2 \le 5 & \\
 & 3X_1 + 2X_2 + 4X_3 + 5Y_1 + 2Y_2 \le 16 & \\
 & \text{- -} & \\
 & 4X_1 + 4X_2 + 5X_3 \le 20 & \\
 & 2X_1 + X_2 \le 4 & \text{(subproblem 1)} \\
 & X_1, X_2, X_3 \ge 0 & \\
 & \text{- -} & \\
 & Y_1 + 0.5Y_2 \le 3 & \\
 & 0.5Y_1 + 0.5Y_2 \le 2 & \text{(subproblem 2)} \\
 & Y_1, Y_2 \ge 0 & \\
 & \text{- -} &
\end{array}
$$

It was decided to use two subproblems, one with variables (X_1,X_2,X_3) and the other with variables (Y_1, Y_2), writing the feasible region of each as a combination of its extreme points.

The tableau below represents the solution of this problem by the Dantzig-Wolfe decomposition algorithm in the midst of the calculations. The variables S_1 and S_2 are slack variables for the first two constraints of the master problem; the variables A_1 and A_2 are artificial variables for the convexity constraints of the master problem.

	λ_1	λ_2	μ_1	μ_2	S_1	S_2	A_1	A_2	RHS
(max)	0	0	0	0	-10	-1	-4	0	1/3
	1	0	0	0	1/3	-1/6	4/3	0	1/3
	0	1	0	0	-1/3	1/6	-1/3	0	2/3
	0	0	1	0	5/12	-1/12	-1/3	0	5/12
	0	0	0	1	-5/12	1/12	1/3	1	7/12

The extreme points (proposals) generated thus far by the subproblems are:

$\underline{X}^1 = (2, 0, 0)$ with weight λ_1

$\underline{X}^2 = (0, 0, 4)$ with weight λ_2

$\underline{Y}^1 = (0, 4)$ with weight μ_1
$\underline{Y}^2 = (0, 0)$ with weight μ_2

(a.) What is the column for ´ $_2$ in the original master tableau?

(b.) What is the current solution (in terms of the X's and Y's) for the master problem?

(c.) What are the values of the simplex multipliers associated with each constraint of the (partial) master problem?

(d.) What are the two subproblems which need to be solved next? (State their objectives and constraints, etc.)

(3.) LP USING LINEAR COMPLEMENTARY SOLUTION TECHNIQUE. Consider the LP problem:

$$
\begin{array}{ll}
\text{MINIMIZE} & 4x_1 + 3x_2 \\
\text{subject to} & 3x_1 + x_2 \geq 10 \\
& 5x_1 + 2x_2 \leq 25 \\
& x_1 - x_2 \geq 8 \\
& x_1 \geq 0, \; x_2 \geq 0
\end{array}
$$

(a.) Write down the primal and the dual problems, both using only equality constraints.

(b.) Write down the tableau containing both sets of equality constraints from part (a.)

(c.) Start with slack and surplus variables in the basis (negating rows with surplus variables as required). Is this solution feasible? Optimal? Does it satisfy complementary slackness?

(d.) Define a single artificial variable and insert its column into the tableau.

(e.) Where should you pivot to enter the artificial variable into the basis? (circle the entry) What variable leaves the basis? Will complementary slackness be satisfied after this pivot?

(f.) What variable or variables may now enter the basis at the next iteration?

(g.) How do you decide when to terminate, since you have no row of reduced costs?

(4.) ANALYSIS OF MPSX OUTPUT: Please refer to your materials on the PURAIR OIL COMPANY (problem statement, formulation, and MPSX output). Answer the following questions (if there is insufficient information in the MPSX output, simply answer NOT ENOUGH INFORMATION):

(a.) Yield structure #2 at refinery 1 is used at full capacity (25000 barrels/day) in the optimal solution. Suppose that it were to be shut down for a portion of a day, so that the capacity that day were only 24500 barrels/day. How much profit would be lost? How would the optimal solution be changed? (refer to the substitution rates)

(b.) HR1M4 denotes the amount of Heavy Fuel Oil produced and shipped to Market #4 from Refinery #1. Suppose that the company has a policy that if a shipment is made, no less than 600 barrels must be shipped. Should the amount shipped be rounded up to 600 or down to zero? How much loss in profit would result? Which nonbasic variable must be adjusted (upward or downward) in order to accomplish this rounding?

(c.) C2M denotes the amount of Crude #2 sold on the open market at $2.45/barrel profit. How low might the profit fall before the solution needs to be adjusted? If the profit reaches this value, how much less will be sold?

(d.) HR2M3 denotes the amount of Heavy Fuel Oil shipped from Refinery #2 to Market #3, at a cost of $0.25/barrel. In the optimal solution, no such shipment is made. By how much must the shipping cost drop before it would be optimal to make this shipment? How much would then be shipped? What other changes would have to be made in the optimal solution?