Name/Initials______________________________

@@@@@ 56:272 Integer Programming & Network Flows @@@@@
@@@@@@@@ Final Exam -- December 16, 1997 @@@@@@@@

Answer **#1** and any **five** of the remaining six problems!

	possible	score
1. Multiple Choice	25	_____
2. Traveling Salesman Problem	15	_____
3. Location in a Network	15	_____
4. Integer LP modeling	15	_____
5. Transportation problem	15	_____
6. Lagrangian relaxation & duality	15	_____
7. Benders' method for Capacitated Plant Location	15	_____

I. Multiple Choice:

1. In a network of **n** nodes and **m** edges, a spanning tree... (check one or more which are true)
   - __a. is strongly connected
   - __b. has **n** nodes
   - __c. has **n** edges
   - __d. *none of the above*
2. An optimal solution of a traveling salesman problem is always...(check one or more)
   - __a. a spanning 1-tree
   - __b. a Hamiltonian tour
   - __c. an Euler tour
   - __d. a feasible solution of an assignment problem
   - __e. *none of the above.*
3. The following problems are in the set P with polynomial-time complexity: (check one or more)
   - __a. minimum spanning tree problem
   - __b. set covering problem
   - __c. (linear) assignment problem
   - __d. generalized assignment problem
   - __e. knapsack problem
   - __f. *none of the above*
4. Dykstra's algorithm...
   - __a. is a special case of the simplex algorithm for linear programming
   - __b. finds a shortest path from a specified source node to each of the other nodes
   - __c. requires a fixed number of iterations.
   - __d. *none of the above.*

___ 5. A minimum-cost network flow problem in a network with **n** nodes and linear costs...
   - a. is an LP problem with **n** basic variables
   - b. has an LP relaxation with a square coefficient matrix
   - c. possesses the "integrality property" if the supplies & demands are integers
   - d. *none of the above.*

___ 6. When applying Benders' method to the capacitated plant location problem, the "master" problem...
   - a. evaluates the total cost if a specified set of plants are open
   - b. selects the next trial set of plants to be open
   - c. gives an upper bound on the cost of the optimal solution
   - d. none of the above.

___ 7. If the current solution of the transportation problem is degenerate...
   - a. the next iteration will produce no improvement in the objective function.
   - b. the reduced cost of at least one zero shipment is zero.
   - c. the number of sources must be equal to the number of destinations.
   - d. *none of the above.*

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___ 8. The *LP relaxation* for an n×n assignment problem...
  a. is guaranteed to have an integer optimal solution only if the costs are integer.
  b. has 2n basic variables.
  c. has only degenerate basic feasible solutions.
  d. may have an optimal cost which is strictly less than the optimal assignment cost.
  e. *none of the above*

___ 9. The quadratic assignment problem...
  a. includes quadratic constraints.
  b. has the same constraints as the linear assignment problem.
  c. includes X_{ij}^2 terms in the objective function.
  d. is a specialized form of the "generalized assignment problem" (GAP).
  e. *none of the above*.

___ 10. The generalized assignment problem...
  a. includes all the constraints of the original assignment problem, plus additional constraints.
  b. can be solved by the Hungarian algorithm together with branch-and-bound
  c. includes the transportation problem as a special case.
  d. *none of the above*.

___ 11. Floyd's algorithm for a graph with n nodes ...
  a. finds the shortest paths from a single source node to each of the other nodes.
  b. requires exactly n iterations to be performed.
  c. is a specialized version of the LP simplex algorithm.
  d. *none of the above*.

___ 12. A node-arc incidence matrix of an undirected graph with n nodes ...
  a. has n rows.
  b. is a square matrix.
  c. has n columns.
  d. *none of the above*.

___ 13. The adjacency matrix of an <u>directed</u> graph with n nodes ...
  a. is a square matrix.
  b. is a symmetric matrix.
  c. has +1, -1, and 0 as entries.
  d. *none of the above*.

___ 14. A minimum spanning tree of an undirected network with n nodes ...
  a. can be given a strongly-connected orientation.
  b. contains no nodes of degree greater than 2
  c. has n-1 edges.
  d. *none of the above*

___ 15. A node-arc incidence matrix of a graph with n nodes ...
  a. has n-1 rows.
  b. is a square matrix.
  c. has rank n-1.
  d. *none of the above*.

___ 16. The adjacency matrix of an <u>undirected</u> graph with n nodes ...
  a. has rank n-1
  b. is a square symmetric matrix.
  c. has a determinant equal to ±1.
  d. *none of the above*.

___ 17. A matrix which is unimodular ...
  a. must be square.
  b. must be lower triangular.
  c. has a determinant equal to zero.
  d. *none of the above*.

___ 18. The vertex penalty method for the traveling salesman problem ...
- a. adds penalties to the vertices within a subtour found by the assignment problem.
- b. is a Lagrangian relaxation of the subtour elimination constraints.
- c. is guaranteed to produce the optimal tour if the optimal penalties are found.
- d. may be used to compute a lower bound for the optimal tour.
- e. *none of the above*

___ 19. Simulated annealing ...
- a. is a type of heuristic algorithm which may be classified as a "contruction" method.
- b. has a parameter referred to as "temperature" which increases at each iteration.
- c. is an iterative algorithm in which the objective function improves at each iteration.
- d. *none of the above*

___ 20. The Lagrangian relaxation of an ILP maximization problem...
- a. has an objective function identical to that of the original problem
- b. has an optimal value which is less than or equal to that of the optimum of the original problem.
- c. has fewer constraints than the original problem.
- d. has an optimal cost which is not lower than that of the LP relaxation.
- e. *none of the above.*

___ 21. The Lagrangian dual of an ILP minimization problem...
- a. provides an upper bound on the optimum of the original problem.
- b. is a maximization problem
- c. has an optimal objective value equal to that of the original problem.
- d. *none of the above.*

II. Traveling Salesman Problem. Five products are to be manufactured weekly on the same machine. The table below gives the cost of switching the machine from one product to another product. (Assume that this is also the cost of switching to the last product of the week to the first product to be scheduled the following week!)

	to:	A	B	C	D	E
from:	A	-	5	7	7	8
	B	3	-	2	8	3
	C	4	2	-	7	3
	D	1	3	3	-	5
	E	5	3	2	6	-

a. The nearest neighbor heuristic, starting with product D, yields the product sequence ___ , ___ , ___ , ___ , ___ , ___ with cost ____.

After row & column reduction of the above matrix to solve the associated assignment problem (with large number, M, inserted along the diagonal), we have:

	to:	A	B	C	D	E
from:	A	M	0	2	2	2
	B	1	M	0	6	0
	C	2	0	M	5	0
	D	0	2	2	M	3
	E	3	3	0	0	M

b. Is there a zero-cost assignment for the above reduced cost matrix? _______

c. If the answer to (b) is "no", perform additional reduction steps as necessary.
What is the solution of this assignment problem?
A-> ___ , B-> ___ , C-> ___ ,D-> ___ ,E-> ___

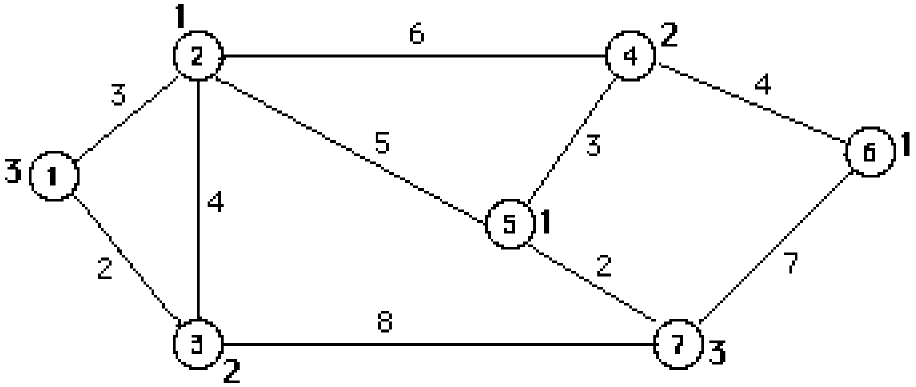
c. What is the cost of this assignment? _________

d. Is it a valid product sequence? _______

e. If yes, is it guaranteed to be optimal? _____

| If not, why not? _________________
| What bound (circle: upper / lower) on the optimal cost does this result provide?
|
| What *single* constraint might be added to the assignment problem which would eliminate the solution which you have obtained (but not eliminate any valid sequence)? *Write the constraint explicitly, giving its coefficients!*
|

III. Location in a Network: Consider the network below, where the bold numbers beside the nodes are the demands to be supplied.



Floyd's algorithm was applied to find the following matrix of shortest path lengths, and the shortest path lengths multiplied by the demand of the destination of each path:

Shortest Path Lengths

to

		1	2	3	4	5	6	7
f	1	0	3	2	9	8	13	10
r	2	3	0	4	6	5	10	7
o	3	2	4	0	10	9	14	8
m	4	9	6	10	0	3	4	5
	5	8	5	9	3	0	7	2
	6	13	10	14	4	7	0	7
	7	10	7	8	5	2	7	0

Weighted Shortest Path Lengths

to

		1	2	3	4	5	6	7
f	1	0	3	2	18	8	26	30
r	2	6	0	4	12	5	20	21
o	3	4	4	0	20	9	28	24
m	4	18	6	10	0	3	8	15
	5	16	5	9	6	0	14	6
	6	26	10	14	8	7	0	21
	7	20	7	8	10	2	14	0

a. Which node is the one-median of this network? ____

b. What is the optimal objective function of the one-median problem? _____

c. What is the objective function value of the **2**-median problem when evaluated at the solution in which facilities are built at nodes 2 and 5? ______

d. Which node is the vertex center of this network (based upon unweighted distances)?

e. Which of the following edge(s) could <u>not</u> have a center which is better than the vertex center? (Indicate with "X"):

[2,5] ___

[2,3] ___

[3,7] ___

IV. Integer LP modeling. Four possible locations of plants are being considered, to supply demand of customers in nine cities. One or more plant locations are to be selected. Let

D_j = annual demand of city j (j=1,2,... 9)
K_i = annual production capacity of plant at location i, *if built* (i=1,2,3,4)
c_{ij} = shipping cost (per unit shipped) between plant i and city j
F_i = annual cost of capital to build & operate a plant at location i

a. Formulate a mixed-integer linear model to choose the locations where plants should be built, using a binary variable Y_i to indicate selection of plant location i.

Minimize

subject to:

How many integer variables are required in this model? _____

How many continuous variables are required in this model? _____

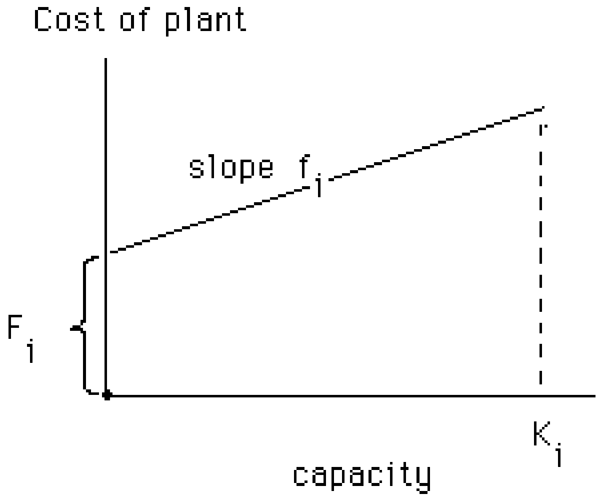
b. Write constraints for each of the additional restrictions:

i.) Plant 4 should not be built if either plant 1 or plant 2 is built.

ii.) No more than 3 plants should be built.

iii) If plant 3 is built, then plant 4 must be built also.

c. Reformulate the model in part (a) so that the capacity of a plant at location i may be any value from zero up to a maximum of K_i, and that the annual cost to build & operate the plant depends upon the capacity which is selected, as indicated in the graph:



d. Reformulate the model in part (c) so that each plant i should not be built unless it is to produce & ship <u>at least</u> a minimum quantity L_i ($0 < L_i < K_i$).

V. Transportation problem: Consider the transportation problem with the tableau below:

destinations

sources	**1**	**2**	**3**	**supply**
1	**10** 9	**2** 5	4	12
2	7	6	12	8
3	10	9	3	10
demand	10	5	15	

a. If the ordinary simplex tableau were to be written for this problem, how many rows (including the objective) will it have? _____

How many columns (excluding the right-hand-side and objective value -z) will it have? ________

b. How many basic variables will this problem have? __________

c. An initial basic feasible solution is found using the "Northwest Corner Method"; complete the computation of this solution and write the values of the variables in the tableau above.

d. If U_1 (the dual variable for the first source) is equal to zero, what is the value of V_3 (the dual variable for the third destination)? ________

e. What is the reduced cost of the variable X_{13}? ___________ (Explain your computation.)

f. Will increasing X_{13} improve the objective function? ___________

g. Regardless of whether the answer to (f) is "yes" or "no", what variable must leave the basis if X_{13} enters?_________

h. What will be the value of X_{13} if it is entered into the solution as in (g)? ___________

VI. Lagrangian Relaxation & Duality: A small airline must schedule a departure from Cedar Rapids Airport to each of four cities: Des Moines, Minneapolis, Chicago, St. Louis. The available departure times are 1 pm, 2 pm, and 3 pm. The airline has only 2 departure lounges, and so at most two flights can be scheduled during a time slot. The airline estimates the following profits per flight (in hundreds of dollars) as a function of departure time:

	Departure time		
Destination	1:00	2:00	3:00
1. Des Moines	10	9	9
2. Minneapolis	11	10	10
3. Chicago	12	10	9
4. St. Louis	10	11	11

Define decision variables

X_{ij} = 1 if flight to destination i is scheduled at time j, and 0 otherwise (i=1,2,3,4; j=1,2,3).

a. Formulate the problem of maximizing profit as an integer LP.

b. Find a feasible solution to the problem. What is its profit? Is this an upper or lower bound on the optimal profit?

c. Apply Lagrangian relaxation to this formulation, relaxing the restriction that no more than two departures may be scheduled for a time slot. Call the three multipliers λ_j, j=1,2, and 3. Write the Lagrangian relaxation:

d. Assign values $\lambda_1=5$, $\lambda_2=4$, and $\lambda_3=3$. What is the solution of the Lagrangian relaxation?

e. What is the bound on the profit provided by this solution of the Lagrangian relaxation? ____
Is it an upper or lower bound? _________

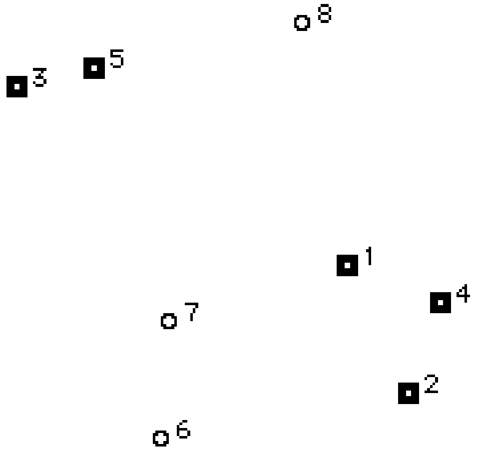
f. Is the solution found in (d) feasible? ____

g. What is the subgradient of the Lagrangian dual objective function evaluated at $\lambda_1=5$, $\lambda_2=4$, and $\lambda_3=3$?
subgradient = (____ , ____ , ____)

h. Determine whether each multiplier should be increased, decreased, or left unchanged in order to improve the bound &/or improve feasibility.

λ_1: __increase, __decrease, __remain unchanged.
λ_2: __increase, __decrease, __remain unchanged.
λ_3: __increase, __decrease, __remain unchanged.

VII. Benders' Decomposition Algorithm for Plant Location: Consider the following randomly-generated problem in which demand in 8 cities is to be satisfied by building plants in one or more of four of the cities:



Costs, Supplies, Demands

i \ j	1	2	3	4	5	6	7	8	K	F
1	0	25	67	18	57	45	34	44	10	5000
2	25	0	89	17	81	45	45	69	17	3000
3	67	89	0	85	14	68	50	52	10	7000
4	18	17	85	0	75	55	49	56	14	4000
5	57	81	14	75	0	67	47	38	18	1000
Demand:	3	5	7	3	8	4	2	9	69	

```
Number of sources      = M = 5
Number of destinations = N = 8
Total demand: 41
```

a. Give the expression for the optimal value as a function of Y, i.e. v(Y), expressed in terms of an optimization of the variables X.

An initial trial solution was evaluated, in which plants 1, 2, 3, & 4 are to be open. The result was:

```
Subproblem Solution

  Plants opened: # 1 2 3 4

Minimum transport cost = 1077
Fixed cost of plants = 19000
       Total = 20077

Generated support is αY+b, where
 α = 5570 4173 7140 4966 1000
 & b = ¯1772

This is support # 1
```

Optimal Shipments

from \ to	1	2	3	4	5	6	7	8	9
1	3	0	0	0	5	0	0	2	0
2	0	5	0	0	0	4	2	0	6
3	0	0	7	0	3	0	0	0	0
4	0	0	0	3	0	0	0	7	4

(Demand pt #9 is dummy demand for excess capacity.)

b. Is the optimal solution of this subproblem degenerate? *Circle:* Yes No
Explain why or why not:

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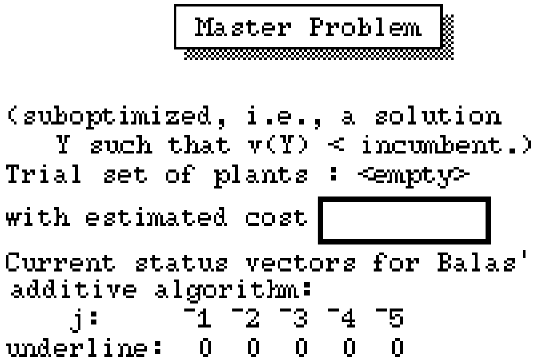
Next a suboptimal solution of the Master Problem is found:

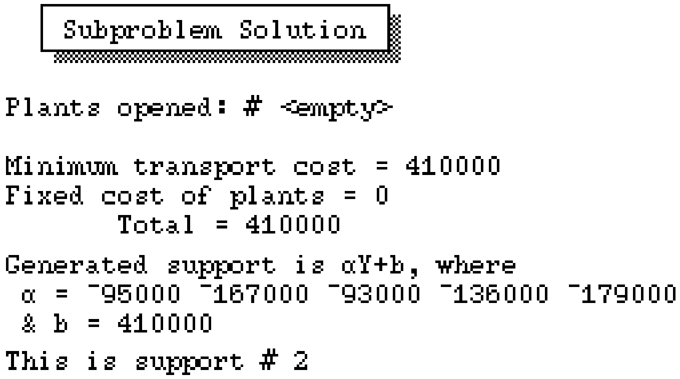


```
Master Problem

(suboptimized, i.e., a solution
   Y such that v(Y) < incumbent.)
Trial set of plants : <empty>
with estimated cost
Current status vectors for Balas'
 additive algorithm:
    j:     ¯1 ¯2 ¯3 ¯4 ¯5
underline:  0  0  0  0  0
```

c. What is the value $\underline{v}_1$(0,0,0,0,0) of the master problem objective which is blanked out above? ________

Using this solution of the master problem (which was sub-optimized), the subproblem, i.e. transportation problem, was next solved:



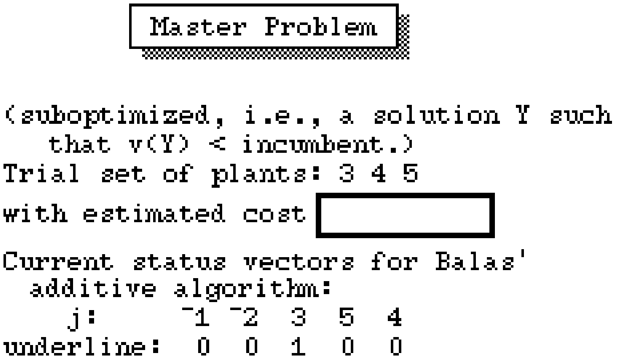
```
Subproblem Solution

Plants opened: # <empty>

Minimum transport cost = 410000
Fixed cost of plants = 0
       Total = 410000
Generated support is αY+b, where
 α = ¯95000 ¯167000 ¯93000 ¯136000 ¯179000
 & b = 410000
This is support # 2
```

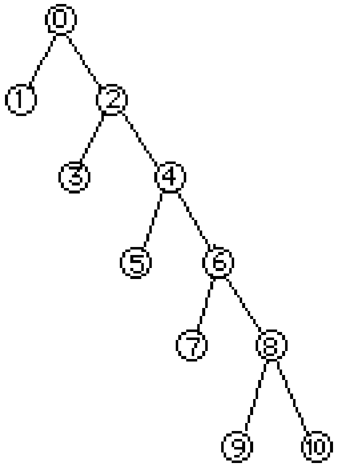
(Note that this subproblem would be infeasible, except that a "dummy" source was included with a "shipping cost" from this source being very high, namely, $10,000 per unit shipped).

Next the master problem is sub-optimized again:



```
Master Problem

(suboptimized, i.e., a solution Y such
   that v(Y) < incumbent.)
Trial set of plants: 3 4 5
with estimated cost
Current status vectors for Balas'
  additive algorithm:
    j:     ¯1 ¯2  3  5  4
underline:  0  0  1  0  0
```



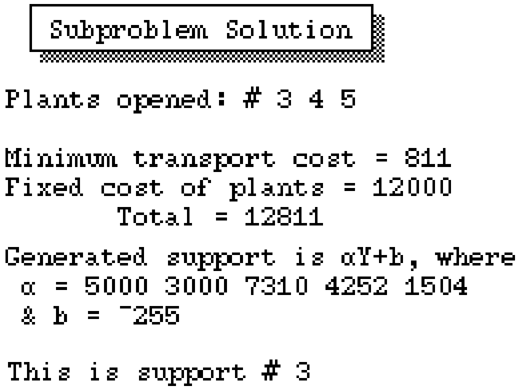
d. What is the value blanked out in the master problem solution above? _________

e. Suppose that node #10 on the implicit enumeration tree above represents this master problem solution. Which nodes have already been fathomed? _________________

f. Which variables have been fixed at node #4? ______________

g. After node #10 is fathomed, which node will be considered next in the implicit enumeration? _______

Next the subproblem was solved, using the set of plants {3, 4, 5}:



```
Subproblem Solution

Plants opened: # 3 4 5

Minimum transport cost = 811
Fixed cost of plants = 12000
       Total = 12811
Generated support is αY+b, where
 α = 5000 3000 7310 4252 1504
 & b = ¯255

This is support # 3
```

h. The third support provides an underestimate of the optimal value function v(Y) which is exactly equal to v(Y) for what values of Y? Y = (___ , ___ , ___ , ___)