Name/Initials_______________________________

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@@ 56:272 Integer Programming & Network Flows @@

@@@@@@@@@@ Final Exam - December 18, 1996 @@@@@@@@@@

| | possible | score |
|---|---|---|
| 1. Multiple Choice | 25 | _____ |
| 2. Traveling Salesman Problem | 15 | _____ |
| 3. Median of Network | 12 | _____ |
| 4. Center of Network | 16 | _____ |
| 5. Primal Simplex Method for Networks | 20 | _____ |
| 6. Project networks | 15 | _____ |
| 7. Generalized Assignment Problem | 12 | _____ |
| 8. Benders' method for Capacitated Plant Location | 20 | _____ |

*(Approximately 15 minutes per problem!)*

**I. Multiple Choice:**

**___ 1. Balas' algorithm is referred to as the "Additive Algorithm" because ...**
- **a. the objective is the sum of (nonnegative) costs.**
- **b. variables are added one at a time to the set of fixed variables.**
- **c. no multiplications or divisions are required**
- **d. none of the above.**

**___ 2. An optimal solution of a traveling salesman problem is always...**
- **a. a Hamiltonial tour**
- **b. a Lagrangian tour**
- **c. an Euler tour**
- **d. none of the above.**

**___ 3. A simple plant location...**
- **a. places no limits on the plant capacities.**
- **b. is also referred to as the median problem.**
- **c. specifies the values of the plant capacities (if built).**
- **d. none of the above.**

**___ 4. When applying Benders' method to the capacitated plant location problem, th problem...**
- **a. evaluates the total cost if a specified set of plants are open**
- **b. selects the next trial set of plants to be open**
- **c. gives an upper bound on the cost of the optimal solution**
- **d. none of the above.**

**___ 5. The "rural" postman problem differs from the original postman problem in t**
- **a. the postman is required to travel only a subset of the total set of edges.**
- **b. the total length of a tour is restricted to a day's travel time.**
- **c. the edges that may be traveled more than once is limited to a subset of the edges.**
- **d. none of the above.**

**___ 6. The quadratic assignment problem...**
- **a. includes quadratic constraints.**
- **b. has the same constraints as the original assignment problem.**
- **c. includes $X_{ij}^2$ terms in the objective function.**
- **d. is a specialized form of the "generalized assignment problem" (GAP).**
- **e. none of the above.**

**___ 7. The generalized assignment problem...**
- **a. includes the original assignment constraints, plus some additional constra**
- **b. can be solved by the Hungarian algorithm together with branch-and-boun**
- **c. includes the transportation problem as a special case.**
- **d. none of the above.**

___ 8. A genetic algorithm for the line-balancing problem with N tasks to be assign
stations ...
  a. if it converges, guarantees that the solution is optimal.
  b. uses a population size equal to N.
  c. represents an individual within the population by a string of numbers of l
  d. none of the above.

___ 9. Floyd's algorithm for a graph with n nodes ...
  a. finds the shortest paths from a single source node to each of the other nod
  b. requires exactly n iterations to be performed.
  c. is a specialized version of the LP simplex algorithm.
  d. none of the above.

___ 10. If the current solution of the transportation problem is degenerate...
  a. the next iteration will produce no improvement in the objective function.
  b. the reduced cost of at least one zero shipment is zero.
  c. the number of sources must be equal to the number of destinations.
  d. none of the above.

___ 11. A node-arc incidence matrix of an undirected graph with n nodes ...
  a. has n rows.
  b. is a square matrix.
  c. has n columns.
  d. none of the above.

___ 12. The adjacency matrix of a <u>directed</u> graph with n nodes ...
  a. is a square matrix.
  b. is a symmetric matrix.
  c. has +1, -1, and 0 as entries.
  d. none of the above.

___ 13. The LP formulation of the problem to find the shortest path in a network ...
  a. has right-hand-sides which are all zero.
  b. may require branch-and-bound if the solution is not integer.
  c. has a dual LP which finds the longest path in a network.
  d. none of the above.

___ 14. The LP model for an nxn assignment problem...
  a. has an integer optimal solution only if the costs are integer.
  b. has 2n basic variables.
  c. has only degenerate basic feasible solutions.
  d. none of the above

___ 15. The following is true of an n-item zero-one knapsack problem with integer
item values, and capacity...
  a. when solving by branch-&-bound, the # of terminal nodes in the <u>complete</u> search
tree is $n^2$
  b. in the DP model, the state variable has $2^n$ possible values
  c. in the DP model, the number of stages is n
  d. none of the above

___ 16. When using the Hungarian method to solve assignment problems, if the nu
lines drawn to cover the zeroes in the reduced matrix is equal to the number of
  a. a mistake has been made, and one should review previous steps.
  b. this indicates that no solution exists.
  c. this means that the current cost matrix has a zero-cost assignment.
  d. none of the above.

___ 17. A minimum spanning tree of an undirected network with n nodes ...
  a. can be given a strongly-connected orientation.
  b. contains no nodes of degree 2
  c. has n-1 edges.
  d. none of the above

___ 18. Vogel's Approximation Method (VAM)...

a. always yields a basic feasible solution of a transportation problem.
b. cannot be applied to an assignment problem, because of degeneracy.
c. will never result in a degenerate solution.
d. none of the above

___ 19. A matrix which is unimodular ...
a. must be square.
b. must be lower triangular.
c. has a determinant equal to zero.
d. none of the above.

___ 20. A node-arc incidence matrix of a graph with n nodes ...
a. has n-1 rows.
b. is a square matrix.
c. has rank n-1.
d. none of the above.

___ 21. The adjacency matrix of an <u>undirected</u> graph with n nodes ...
a. has rank n-1
b. is a square symmetric matrix.
c. has a determinant equal to ±1
d. none of the above.

___ 22. The LP formulation of the problem to find the minimum completion time of a
a. has two variables for each project activity.
b. may require branch-and-bound if the solution is not integer.
c. has a dual LP which finds the longest path in the project network.
d. none of the above.

___ 23. Djikstra's algorithm for a graph with n nodes ...
a. finds the shortest paths from a source node to each of the other nodes.
b. requires exactly n iterations to be performed.
c. is a special case of the simplex algorithm for LP.
d. none of the above.

___ 24. The vertex penalty method for the traveling salesman problem ...
a. adds penalties to the vertices within a subtour found by the assignment pr
b. is an example of Lagrangian relaxation.
c. may be used to compute an upper bound.
d. none of the above.

___ 25. In simulated annealing, ...
a. the probability of accepting a worse solution decreases at each iteration.
b. no initial feasible solution is required.
c. the objective function improves at each iteration.
d. none of the above.

**II. Traveling Salesman Problem.** Five products are to be manufactured weekly on the same machine. The table below gives the cost of switching the machine from one product to another product. (Assume that this is also the cost of switching to th last product of the week to the first product to be scheduled the following week!)

| from: \ to: | A | B | C | D | E |
|---|---|---|---|---|---|
| A | - | 6 | 7 | 7 | 6 |
| B | 3 | - | 2 | 8 | 3 |
| C | 4 | 2 | - | 7 | 3 |
| D | 1 | 3 | 3 | - | 5 |
| E | 5 | 3 | 2 | 6 | - |

a. The nearest neighbor heuristic, starting with product D, yields the product sequence ___ , ___ , ___ , ___ , ___ , ___ with cost ____.

Name/Initials______________________________

**After row & column reduction of the above matrix to solve the associated assignme problem (with large number, M, inserted along the diagonal), we have:**

| to: | A | B | C | D | E |
|---|---|---|---|---|---|
| from A | M | 1 | 2 | 2 | 0 |
| B | 1 | M | 0 | 6 | 1 |
| C | 2 | 0 | M | 5 | 1 |
| D | 0 | 2 | 2 | M | 3 |
| E | 3 | 3 | 0 | 0 | M |

**b. Is there a zero-cost assignment for the above reduced cost matrix? _______**

**c. If the answer to (b) is "no", perform additional reduction steps as necessar What is the solution of this assignment problem?**
**A-> ___ , B-> ___ , C-> ___ ,D-> ___ ,E-> ___**

**c. What is the cost of this assignment? _________**

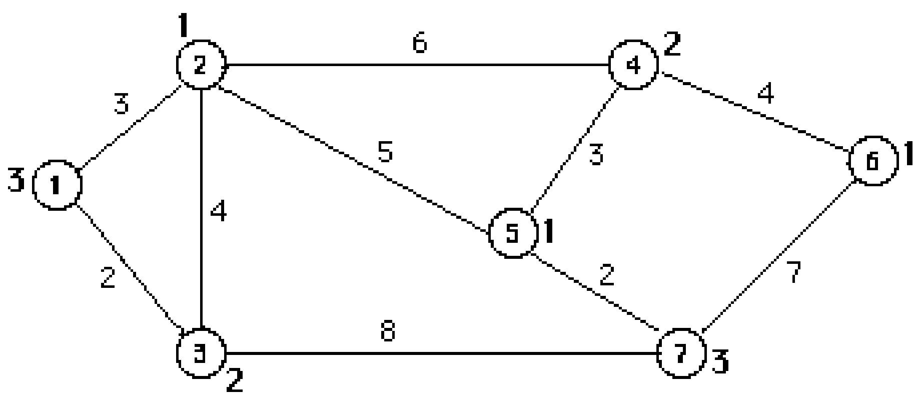
**d. Is it a valid product sequence? _______**

e. If yes, is it guaranteed to be optimal? _____

| If not, why not? _________________
| What bound (circle: upper / lower ) on the optimal cost does this result provide?

| What single constraint might be added to the assignment problem which would eliminate the solution which you have obtained (but not eliminate any valid sequence)?

**III. The Median Plant Location Problem: Consider the network below, where the bold numbers beside the nodes are the demands to be supplied.**




**Floyd's algorithm was applied to find the following matrix of shortest path lengths:**

Name/Initials_______________________________

**Shortest Path Lengths**

| from \ to | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 3 | 2 | 9 | 8 | 13 | 10 |
| 2 | 3 | 0 | 4 | 6 | 5 | 10 | 7 |
| 3 | 2 | 4 | 0 | 10 | 9 | 14 | 8 |
| 4 | 9 | 6 | 10 | 0 | 3 | 4 | 5 |
| 5 | 8 | 5 | 9 | 3 | 0 | 7 | 2 |
| 6 | 13 | 10 | 14 | 4 | 7 | 0 | 7 |
| 7 | 10 | 7 | 8 | 5 | 2 | 7 | 0 |

**Weighted Shortest Path Lengths**

| from \ to | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 3 | 2 | 18 | 8 | 26 | 30 |
| 2 | 6 | 0 | 4 | 12 | 5 | 20 | 21 |
| 3 | 4 | 4 | 0 | 20 | 9 | 28 | 24 |
| 4 | 18 | 6 | 10 | 0 | 3 | 8 | 15 |
| 5 | 16 | 5 | 9 | 6 | 0 | 14 | 6 |
| 6 | 26 | 10 | 14 | 8 | 7 | 0 | 21 |
| 7 | 20 | 7 | 8 | 10 | 2 | 14 | 0 |

**The addition/substitution heuristic was applied to try to find the 2-median set, giving the output below:**

**K-median Facility Location Problem**

1-Median:

Location 5

Cost = **a**

2-Median:

Trial additions:

| Add | 1 | 2 | 3 | 4 | 6 | 7 |
|---|---|---|---|---|---|---|
| cost | 31 | 36 | 34 | 44 | 42 | 49 |

Addition result: Locations 5 & **b**

Cost: **c**

Substitution Step

| Cost | Locations |
|---|---|
| 60 | 2 1 |
| 36 | 5 2 |
| 79 | 3 1 |
| 34 | 5 3 |
| 31 | 4 1 |
| 44 | 5 4 |
| 41 | 6 1 |
| 42 | 5 6 |
| 31 | 7 1 |
| 49 | 5 7 |

Substitution result: Locations **d** & **e**

Cost: **f**

**Six values are blanked in the output of the addition/substitution heuristic. Wha are these values?**

**a. __________ (the cost of the 1-median set {5})**

**b. __________ (the facility added to the 1-median set {5} )**

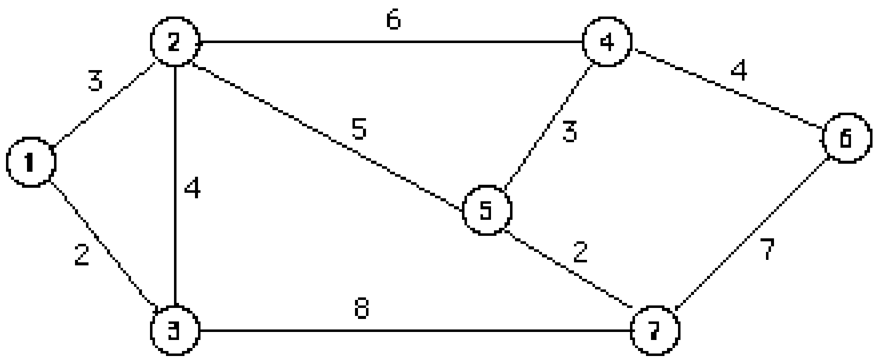
**c. __________ (the cost of the solution after the addition step)**

**d & e. _________ (the pair of facilities resulting from the substitution step**

**f. __________ (the cost of the final solution)**

**IV. Center of Network. Consider the network :**

Name/Initials________________________________




Shortest Path Lengths

to / from

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 3 | 2 | 9 | 8 | 13 | 10 |
| 2 | 3 | 0 | 4 | 6 | 5 | 10 | 7 |
| 3 | 2 | 4 | 0 | 10 | 9 | 14 | 8 |
| 4 | 9 | 6 | 10 | 0 | 3 | 4 | 5 |
| 5 | 8 | 5 | 9 | 3 | 0 | 7 | 2 |
| 6 | 13 | 10 | 14 | 4 | 7 | 0 | 7 |
| 7 | 10 | 7 | 8 | 5 | 2 | 7 | 0 |

**a. Find the vertexcenter for the network. _______**

**b. Below is some output displaying a lower bound which may be computed for th center objective function on each edge. What is the missing value? _______**

Lower Bounds on Local Edge-Centers

| i | j | LB |
|---|---|---|
| 1 | 2 | 10 |
| 1 | 3 | 12.5 |
| 2 | 3 | 10 |
| 2 | 4 | 7 |
| 2 | 5 | 7 |
| 3 | 7 | 8 |
| 4 | 5 | |
| 4 | 6 | 10 |
| 5 | 7 | 8.5 |
| 6 | 7 | 8.5 |

**d. Based on (c), which edges can be eliminated from consideration when searching for the absolute center? __________________________**

**e. Below is information about the center objective function on the edge (4,5). What are the three missing values?**

Minimax Objective on Edge (4,5)

Monotonically increasing distance functions: d(x,k) where

| | | |
|---|---|---|
| k= | 4 | 6 |
| d(i,k)= | 0 | 4 |
| d(j,k)= | 3 | a |

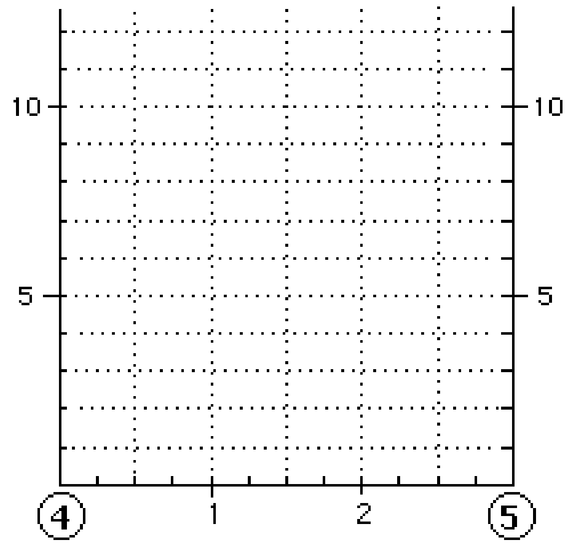
Monotonically decreasing distance functions: d(x,k) where

| | | |
|---|---|---|
| k= | 5 | 7 |
| d(i,k)= | 3 | 5 |
| d(j,k)= | 0 | 2 |

Distance functions which increase to a peak at a point Δ units from i, then decrease: d(x,k) where

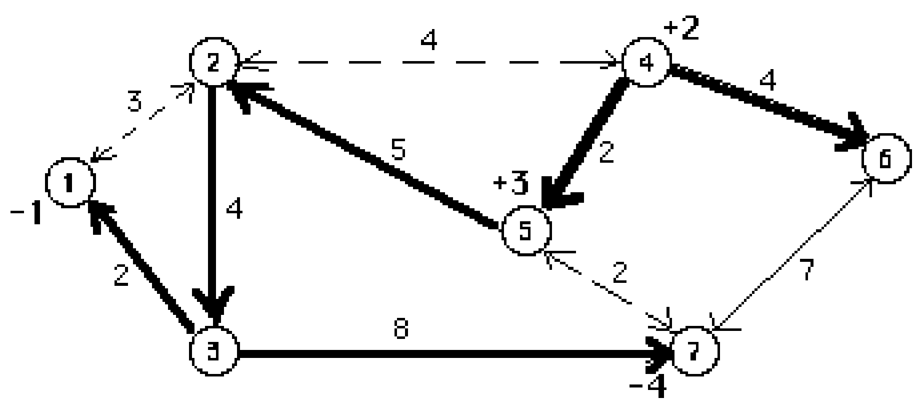
| | | | |
|---|---|---|---|
| k= | 1 | 2 | 3 |
| d(i,k)= | 9 | 6 | b |
| d(j,k)= | 8 | 5 | 9 |
| Δ= | c | 1 | 1 |

**h. Sketch the center objective function on the edge (4,5). What is the edge center of the edge (4,5)?**




**V. Primal Simplex Algorithm for Networks.**

**Consider the network below, where the number alongside each node represents supply or demand, i.e., node #4 has a supply of 2 units of a commodity, node #5 has 3 unit, node #1 requires 1 unit, and node #7 requires 4 units. The numbers alongside the arcs represent unit shipping costs. The initial feasible solution is shown in bol**



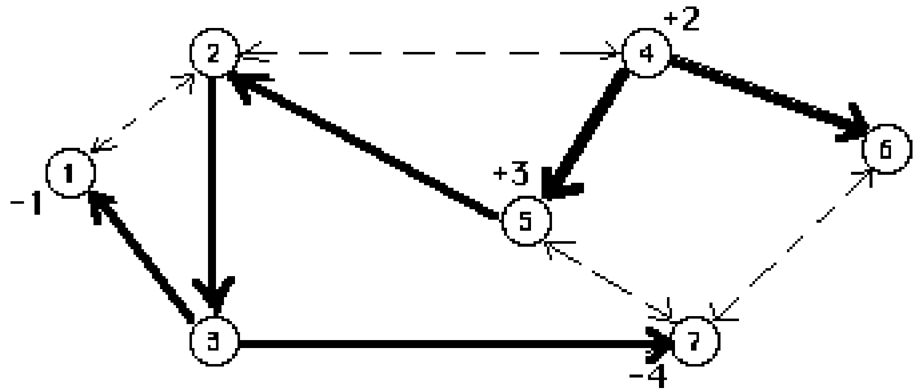

**a. The node-arc incidence matrix will have ____ rows and _____ columns.**

**b. In order to obtain a complete basis of the LP, an "artificial" arc must be added. Indicate it above by adding an arc.**

**c. In the LP to find the minimum-cost flow, how many rows are there in the constraint matrix? _____ How many columns? _______**

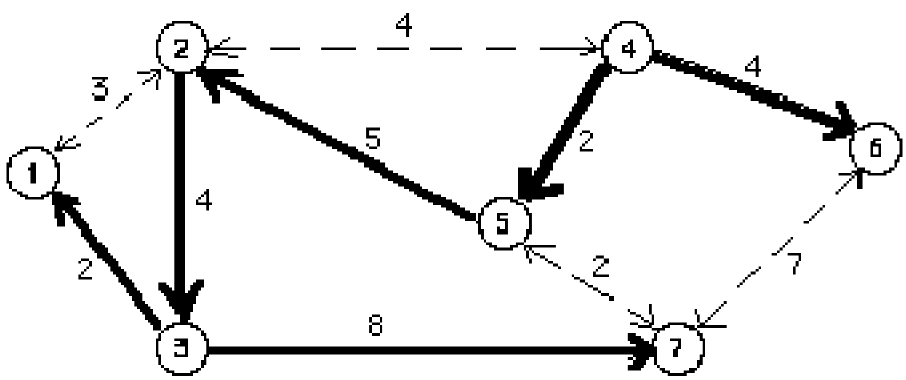
**d. Write the node-arc incidence matrix of the subgraph representing the above basis of the LP.**

**e. Using the minimum spanning tree (plus artificial "root" arc) as an initial basi compute the corresponding basic solution, i.e., flows. Indicate these flows belo on the arcs:**



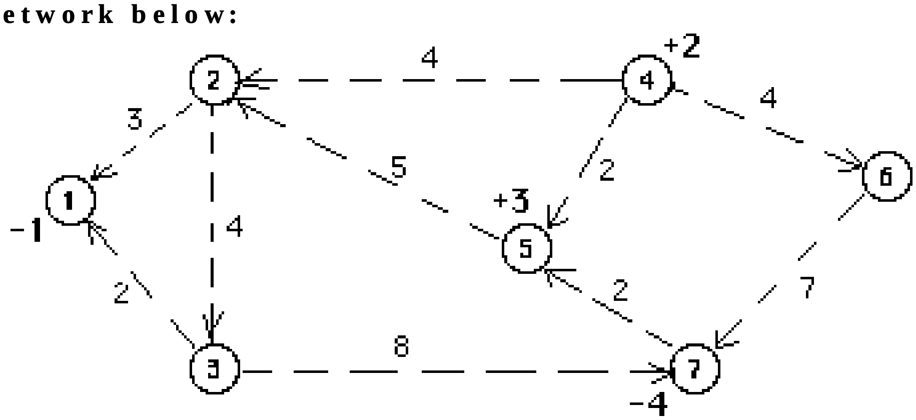

**f. Is the basic solution in (e) degenerate? _____**

**g. Using the same basis, compute the dual variables (simplex multipliers), and indicate below, alongside each node:**

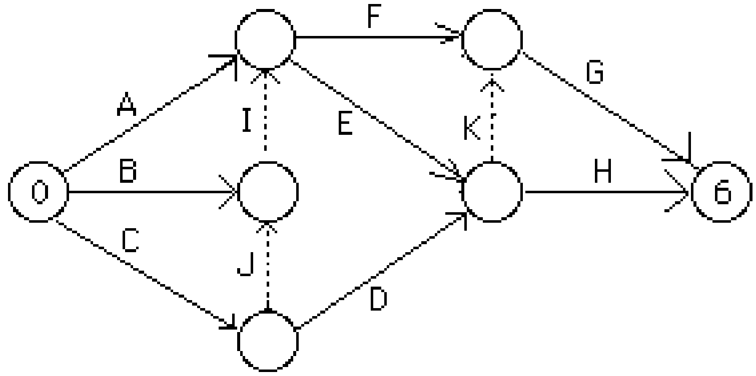



**h. "Price" the arc (6,7), i.e., compute its reduced cost. Should this arc enter the basis or not?**

**i. Regardless of whether the above test indicates that the arc (6,7) should enter t basis, please enter that arc into the basis and indicate the new basis on the network below:**



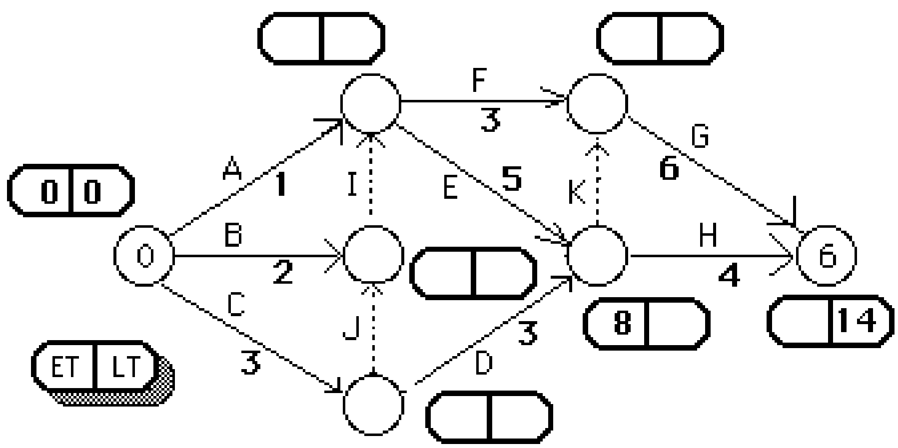
**VI. Project Scheduling.** **Consider the project with the A-O-A (activity-on-arrow) n given below.**




**a. How many activities (i.e., tasks), including "dummies", are required to complete th project? _________**

**b. Complete the labeling of the nodes 1,2,3,4, &5 on the network above.**

**c. The activity durations are given below on the arrows. Compute the Early Times ( Late Times (LT) for each node, writing them in the box (with rounded corners) b node.**

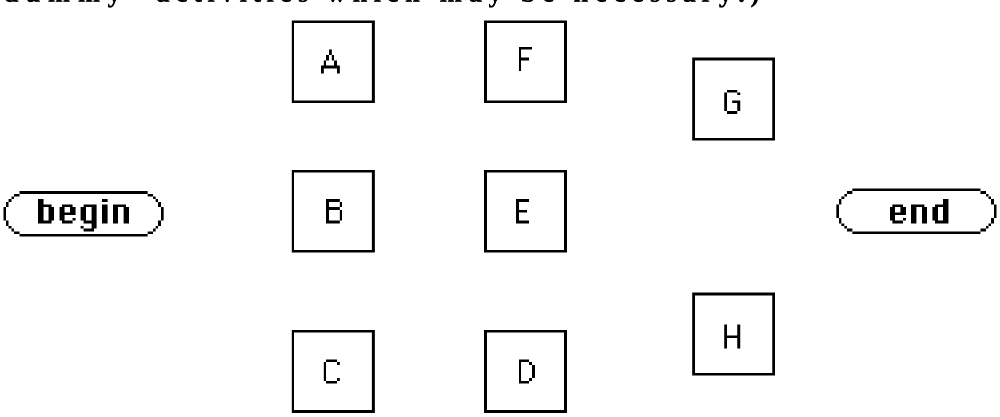



**d. Find the slack ("total float") for activity F. __________**

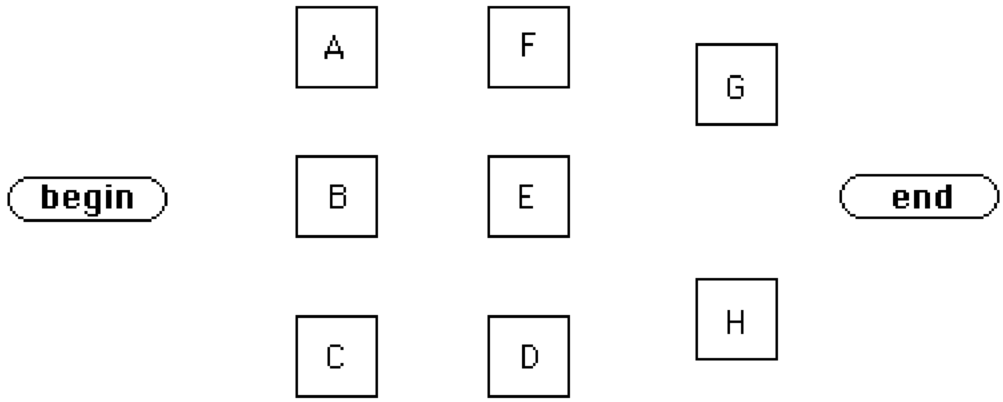
**e. Which activities are critical? (circle: A B C D E F G H I J K )**

**f. What is the earliest completion time for the project? ____________**

**g. Complete the A-O-N (activity-on-node) network below for this same project. (Ad "dummy" activities which may be necessary.)**



**h. Suppose that the dummy activity labelled "J" is deleted. Indicate the resulting network below:**




**VII. Generalized Assignment Problem: Consider the problem of assigning 6 jobs to 3 machines (each with limited capacity):**

Resources Used

| Machine i | Job 1 | Job 2 | Job 3 | Job 4 | Job 5 | Job 6 | Available b |
|---|---|---|---|---|---|---|---|
| 1 | 18 | 22 | 24 | 21 | 24 | 16 | 20 |
| 2 | 23 | 21 | 18 | 15 | 24 | 6 | 27 |
| 3 | 15 | 21 | 14 | 15 | 13 | 14 | 38 |

Costs

| Machine i | Job 1 | Job 2 | Job 3 | Job 4 | Job 5 | Job 6 |
|---|---|---|---|---|---|---|
| 1 | 10 | 18 | 11 | 17 | 24 | 13 |
| 2 | 11 | 20 | 25 | 24 | 18 | 13 |
| 3 | 23 | 11 | 17 | 18 | 18 | 19 |

**a. Formulate this problem as a binary integer programming problem.**

**b. Suppose that the machine capacity constraints are relaxed, using the Lagrangia relaxation method. The first 2 iterations of the subgradient optimization method to maximize the lower bound appears below.**

```
Lambda = 0.75
Upper bound Z* = 120
                    Iteration # 1

Multiplier vector U = 0 0 0
Objective function of relaxation:
```

| machine \ job | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 10 | 18 | 11 | 17 | 24 | 13 |
| 2 | 11 | 20 | 25 | 24 | 18 | 13 |
| 3 | 23 | 11 | 17 | 18 | 18 | 19 |

```
Dual value is [ A ]
Variables selected from GUB sets are:
    1 3 1 1 2 1
Resources used are: 79 24 21, (Available: 20 27 38)
Subgradient of Dual Objective is [ B | C | D ]
Stepsize is 0.00861821
```

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**Iteration # 2**

Multiplier vector U = 0.508475 0 0

Objective function of relaxation:

| machine \ job | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 19.15 | 29.18 | 23.20 | 27.67 | 36.20 | 21.13 |
| 2 | 11 | 20 | 25 | 24 | 18 | 13 |
| 3 | 23 | 11 | 17 | 18 | 18 | 19 |

Dual value is 77.8305

Variables selected from GUB sets are:

| E | F | G | H | I | J |
|---|---|---|---|---|---|

Resources used are: 0 53 50, (Available: 20 27 38)

Subgradient of Dual Objective is ¯20 26 12

Stepsize is [ K ]

**c. Several values have been omitted from the output. Compute their values (not including the stepsize K!):**

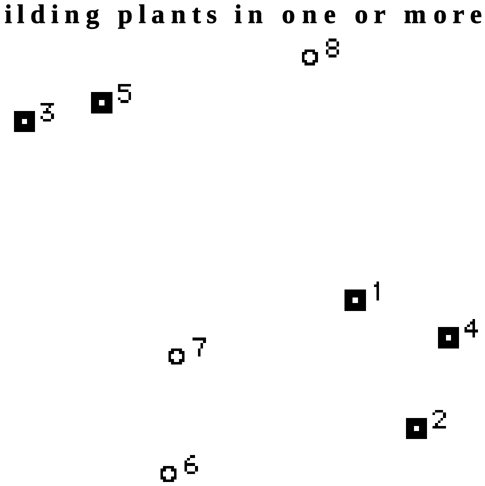
**A** _____ **B** _____ **C** _____ **D** ______ **E** _____

**F** _____ **G** _____ **H** _____ **I** ______ **J** _____

**d. Does this Lagrangian relaxation possess the "integrality property"? Circle: Yes No**
**Why or why not?**

**e. Based upon your answer in (d), the lower bound which can be obtained from this relaxation is (circle: ≤, =, ≥) that of the LP relaxation.**

**VIII. Benders' Decomposition Algorithm for Plant Location: Consider the following randomly-generated problem in which demand in 8 cities is to be satisfied by building plants in one or more of four of the cities:**




**Costs, Supplies, Demands**

| i \ j | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | K | F |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 25 | 67 | 18 | 57 | 45 | 34 | 44 | 10 | 5000 |
| 2 | 25 | 0 | 89 | 17 | 81 | 45 | 45 | 69 | 17 | 3000 |
| 3 | 67 | 89 | 0 | 85 | 14 | 68 | 50 | 52 | 10 | 7000 |
| 4 | 18 | 17 | 85 | 0 | 75 | 55 | 49 | 56 | 14 | 4000 |
| 5 | 57 | 81 | 14 | 75 | 0 | 67 | 47 | 38 | 18 | 1000 |
| Demand: | 3 | 5 | 7 | 3 | 8 | 4 | 2 | 9 | 69 | |

Number of sources = M = 5
Number of destinations = N = 8
Total demand: 41

**a. State the mixed-integer programming formulation of the problem. How many continuous variables (X) and how many binary (zero-one) variables (Y) are required?**

**A trial solution was evaluated, in which plants 1, 2, 3, & 4 are to be open. The result was:**

Subproblem Solution

Plants opened: # 1 2 3 4

Minimum transport cost = 1077
Fixed cost of plants = 19000
Total = 20077

Generated support is αY+b, where
α = 5570 4173 7140 4966 1000
& b = ¯1772

This is support # 1

Optimal Shipments

| from \ to | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 0 | 0 | 0 | 5 | 0 | 0 | 2 | 0 |
| 2 | 0 | 5 | 0 | 0 | 0 | 4 | 2 | 0 | 6 |
| 3 | 0 | 0 | 7 | 0 | 3 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 3 | 0 | 0 | 0 | 7 | 4 |

(Demand pt #9 is dummy demand for excess capacity.)

**b. Is the optimal solution of this subproblem degenerate? Circle: Yes No**
**Why or why not?**

**Next a suboptimal solution of the Master Problem is found:**

Master Problem

(suboptimized, i.e., a solution
Y such that v(Y) < incumbent.)
Trial set of plants : <empty>
with estimated cost [ ]

Current status vectors for Balas' additive algorithm:

| j: | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 |
|---|---|---|---|---|---|
| underline: | 0 | 0 | 0 | 0 | 0 |

**c. What is the value $\underline{v}$(0,0,0,0,0) of the master problem objective which is blanked out above?**

**Using this solution of the master problem (which was sub-optimized), the subproblem, i.e. transportation problem, was next solved:**

Subproblem Solution

Plants opened: # <empty>

Minimum transport cost = 410000
Fixed cost of plants = 0
Total = 410000

Generated support is αY+b, where
α = ¯95000 ¯167000 ¯93000 ¯136000 ¯179000
& b = 410000

This is support # 2

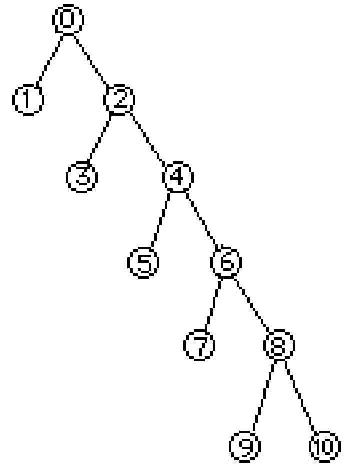
**Next the master problem is sub-optimized again:**

**Master Problem**

(suboptimized, i.e., a solution Y such
that v(Y) < incumbent.)
Trial set of plants: 3 4 5
with estimated cost [ ]

Current status vectors for Balas' additive algorithm:

| j: | ¯1 | ¯2 | 3 | 5 | 4 |
|---|---|---|---|---|---|
| underline: | 0 | 0 | 1 | 0 | 0 |




**d. What is the value blanked out in the master problem solution above?**

**e. This is accepted by the Master problem because $v_2(0,0,1,1,1)$ is less than the incumbent value, which is , ____________.**

**f. Suppose that node #10 on the implicit enumeration tree above represents the master problem solution. Which nodes have already been fathomed? ______________**

**g. Which variables have been fixed at node #6? _______________**

**h. After node #10 is fathomed, which node is considered next in the implicit enumeration? _______**

**Next the subproblem was solved, using the set of plants {3, 4, 5}:**

**Subproblem Solution**

Plants opened: # 3 4 5

Minimum transport cost = 811
Fixed cost of plants = 12000
Total = 12811

Generated support is αY+b, where
α = 5000 3000 7310 4252 1504
& b = ¯255

This is support # 3

**i. Suppose that, based upon the approximation $v_3(Y)$ which we have constructed, we wish to estimate the cost of the proposal to open plants #1, 2, &4.**

**Current List of Supports of v(Y)**

Open plants: 1 2 4

Current approximation of v(Y) is
Maximum { α[i]Y + b[i] }
where α & b are:

| $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ | $\alpha_5$ | b |
|---|---|---|---|---|---|
| 5570 | 4173 | 7140 | 4966 | 1000 | ¯1772 |
| ¯95000 | ¯167000 | ¯93000 | ¯136000 | ¯179000 | 410000 |
| 5000 | 3000 | 7310 | 4252 | 1504 | ¯255 |

| support k | $\sum_{i=1}^{5} \alpha_i^k Y_i + \beta^i$ |
|---|---|
| 1 | 12937 |
| 2 | 12000 |
| 3 | 11977 |

**What is the value of** $V_3(1,1,0,1,0)$? ____________
**Does this give us an over- or under-estimate of the cost?** ________
**Could the set of plants {1,2,4} possibly be optimal?** _____
**Explain your answer:**