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56:272 Integer Programming & Network Flows
Final Exam - December 13, 1995

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Select a total of five problems:

1. Knapsack problem
2. Generalized Assignment Problem
3. Benders' decomposition
4. Integer Programming Model Formulation
5. Traveling Salesman Problem
6. Lagrangian relaxation

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1. Knapsack Problem: A "knapsack" is to be filled to maximize the value of the contents, subject to a weight restriction:

i	W	V
1	3	4
2	9	5
3	6	6
4	4	3
5	3	2

Sorted by Value /Weight

i	W	V	V/W	r
1	3	4	1.33333	1
3	6	6	1	2
4	4	3	0.75	3
5	3	2	0.666667	4
2	9	5	0.555556	5

```
W = 'weight' of item     V/W = value per unit weight
V = value of item        r = rank by V/W
```

At most one unit of an item should be included. The total weight of the knapsack cannot exceed 12 pounds.

a. Formulate this problem as a 0-1 ILP problem.

Partial output of the branch-&-bound algorithm for this problem appears below:

```
>>>Subproblem # 1
Forced in:
Forced out:
Free:        1  2  3  4  5
Fractional solution: selected items = 1 3
                     plus 0.75 of item # 4
                     value = 12.25
Rounding down yields value 10
*** NEW INCUMBENT! ***
    >>>Subproblem # 2
    Forced in:    4
    Forced out:
    Free:        1  2  3  5
    Fractional solution: selected items = 1 4
                         plus 0.833333 of item # 3
                         value = 12
    Rounding down yields value 7
```

```
>>>Subproblem # 3
Forced in:     3  4
Forced out:
Free:          1  2  5
Fractional solution: selected items = 3 4
                    plus 0.666667 of item # 1
                    value = 11.6667
Rounding down yields value 9
   >>>Subproblem # 4
   Forced in:     1  3  4
   Forced out:
   Free:          2  5
   Infeasible!
   <<<Subproblem # 4 fathomed.
   >>>Subproblem # 5
   Forced in:     3  4
   Forced out:    1
   Free:          2  5
   Fractional solution: selected items = [__________]
                       plus [______] of item # [__]
                       value = [______]
   Rounding down yields value [____]
```

b. Complete the blanks in the output above.

c. Draw the current search tree, using the information in the above output.

Dynamic programming output for this problem is given below:

STAGE 1

s \ x:	0	1
0	0.00	¯999999.00
1	0.00	¯999999.00
2	0.00	¯999999.00
3	0.00	4.00
4	0.00	4.00
5	0.00	4.00
6	0.00	4.00
7	0.00	4.00
8	0.00	4.00
9	0.00	4.00
10	0.00	4.00
11	0.00	4.00
12	0.00	4.00

STATE	OPTIMAL VALUES	OPTIMAL DECISIONS	RESULTING STATE
0	0.00	0	0
1	0.00	0	1
2	0.00	0	2
3	4.00	1	0
4	4.00	1	1
5	4.00	1	2
6	4.00	1	3
7	4.00	1	4
8	4.00	1	5
9	4.00	1	6
10	4.00	1	7
11	4.00	1	8
12	4.00	1	9

STAGE 2

s \ x:	0	1
0	0.00	¯999999.00
1	0.00	¯999999.00
2	0.00	¯999999.00
3	4.00	¯999999.00
4	4.00	¯999999.00
5	4.00	¯999999.00
6	4.00	¯999999.00
7	4.00	¯999999.00
8	4.00	¯999999.00
9	4.00	5.00
10	4.00	5.00
11	4.00	5.00
12	4.00	9.00

STATE	OPTIMAL VALUES	OPTIMAL DECISIONS	RESULTING STATE
0	0.00	0	0
1	0.00	0	1
2	0.00	0	2
3	4.00	0	3
4	4.00	0	4
5	4.00	0	5
6	4.00	0	6
7	4.00	0	7
8	4.00	0	8
9	5.00	1	0
10	5.00	1	1
11	5.00	1	2
12	9.00	1	3

STAGE 3

s \ x:	0	1
0	0.00	¯999999.00
1	0.00	¯999999.00
2	0.00	¯999999.00
3	4.00	¯999999.00
4	4.00	¯999999.00
5	4.00	¯999999.00
6	4.00	6.00
7	4.00	6.00
8	4.00	6.00
9	5.00	10.00
10	5.00	10.00
11	5.00	10.00
12	9.00	10.00

STATE	OPTIMAL VALUES	OPTIMAL DECISIONS	RESULTING STATE
0	0.00	0	0
1	0.00	0	1
2	0.00	0	2
3	4.00	0	3
4	4.00	0	4
5	4.00	0	5
6	6.00	1	0
7	6.00	1	1
8	6.00	1	2
9	10.00	1	3
10	10.00	1	4
11	10.00	1	5
12	10.00	1	6

STAGE 4

s \ x:	0	1
0	0.00	¯999999.00
1	0.00	¯999999.00
2	0.00	¯999999.00
3	4.00	¯999999.00
4	4.00	3.00
5	4.00	3.00
6	6.00	3.00
7	6.00	7.00
8		
9	10.00	7.00
10	10.00	9.00
11	10.00	9.00
12	10.00	9.00

STATE	OPTIMAL VALUES	OPTIMAL DECISIONS	RESULTING STATE
0	0.00	0	0
1	0.00	0	1
2	0.00	0	2
3	4.00	0	3
4	4.00	0	4
5	4.00	0	5
6	6.00	0	6
7	7.00	1	3
8			
9	10.00	0	9
10	10.00	0	10
11	10.00	0	11
12	10.00	0	12

STAGE 5

s \ x:	0	1
0	0.00	¯999999.00
1	0.00	¯999999.00
2	0.00	¯999999.00
3	4.00	2.00
4	4.00	2.00
5	4.00	2.00
6	6.00	6.00
7	7.00	6.00
8	7.00	6.00
9	10.00	8.00
10	10.00	9.00
11	10.00	9.00
12	10.00	12.00

STATE	OPTIMAL VALUES	OPTIMAL DECISIONS	RESULTING STATE
0	0.00	0	0
1	0.00	0	1
2	0.00	0	2
3	4.00	0	3
4	4.00	0	4
5	4.00	0	5
6	6.00	0	6
		1	3
7	7.00	0	7
8	7.00	0	8
9	10.00	0	9
10	10.00	0	10
11	10.00	0	11
12	12.00	1	9

d. Complete the computations in the five blank boxes in stage 4 of the DP output above.

e. Suppose that the capacity of the knapsack is only 11 pounds. Show how to use the DP output to determine the optimal knapsack contents.

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2. Generalized Assignment Problem: Consider the problem of assigning 6 jobs to 3 machines (each with limited capacity):

Resources Used

Machine i	Job 1	Job 2	Job 3	Job 4	Job 5	Job 6	Available b
1	18	22	24	21	24	16	20
2	23	21	18	15	24	6	27
3	15	21	14	15	13	14	38

Costs

Machine i	Job 1	Job 2	Job 3	Job 4	Job 5	Job 6
1	10	18	11	17	24	13
2	11	20	25	24	18	13
3	23	11	17	18	18	19

a. Formulate this problem as a binary integer programming problem.

b. Suppose that the integer restrictions are relaxed and the problem solved by the simplex LP algorithm. Will the optimal values of the variables be necessarily integer?

c. Suppose that the machine capacity constraints are relaxed, using the Lagrangian relaxation method. The first 2 iterations of the subgradient optimization method to maximize the lower bound appears below, where the optimal value was estimated to be 120, and a stepsize parameter was assigned the value 0.75.

Lambda = 0.75
Upper bound Z* = 120

Iteration # 1

Multiplier vector U = 0 0 0
Objective function of relaxation:

machine \ job	1	2	3	4	5	6
1	10	18	11	17	24	13
2	11	20	25	24	18	13
3	23	11	17	18	18	19

Dual value is [A]
Variables selected from GUB sets are:
1 3 1 1 2 1
Resources used are: 79 24 21, (Available: 20 27 38)
Subgradient of Dual Objective is [B | C | D]
Stepsize is 0.00861821

Iteration # 2

Multiplier vector U = 0.508475 0 0
Objective function of relaxation:

machine \ job	1	2	3	4	5	6
1	E	29.18	23.20	27.67	36.20	21.13
2	11	20	25	24	18	13
3	23	11	17	18	18	19

Dual value is 77.8305
Variables selected from GUB sets are:
[F | G | H | I | J | K]
Resources used are: 0 53 50, (Available: 20 27 38)
Subgradient of Dual Objective is ¯20 26 12
Stepsize is [L]

d. Several values have been omitted from the output. ("Variables selected from GUB sets" refers to the machine selected for each of the jobs.) Compute their values:

A _____ B _____ C _____ D ______ E _____
F _____ G _____ H _____ I ______ J _____
K _____

e. What is the "integrality property" of a Lagrangian relaxation?

Does this particular Lagrangian relaxation have this property? *Circle:* Yes No

f. What does your answer in (e) imply about the strength of the lower bound which can be obtained from this relaxation, compared to that of the LP relaxation? (Is it stronger,weaker, or identical?)

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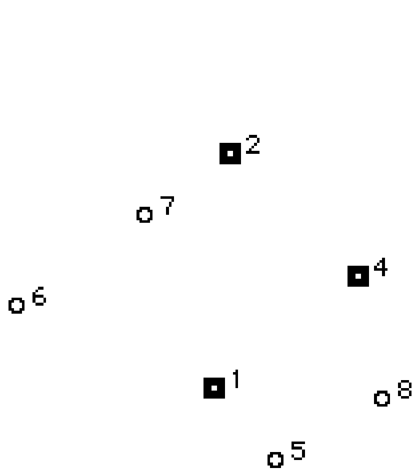
3. Benders' Decomposition of Capacitated Plant Location Problem

Consider the following problem in which demand in 8 cities is to be satisfied by plants to be built in one or more of cities 1,2,3, & 4:




```
Number of sources      = M = 4
Number of destinations = N = 8
Total demand: 40

K = capacity,
F = fixed cost
```

Costs, Supplies, Demands

i\j	1	2	3	4	5	6	7	8	K	F
1	0	42	75	33	17	38	33	30	19	498
2	42	0	43	32	56	47	19	52	15	23
3	75	43	0	46	82	90	62	67	17	89
4	33	32	46	0	36	61	40	22	17	129
Demand	6	1	2	5	5	8	3	10	68	

a. State the mixed-integer programming formulation of the problem. How many continuous variables (X) and how many binary (zero-one) variables (Y) are required?

A trial solution was evaluated, in which all four plants are to be open. The result was:

```
Subproblem Solution

Plants opened: # 1 2 3 4

Minimum transport cost = 666
Fixed cost of plants = 739
       Total = 1405
```

```
Generated support is αY+b, where
 α = 498 23 89 129
 & b = 666
That is, v(Y) ≥ αY+b

This is support # 1
```

The Master Problem was next optimized. (A constraint $\sum_{i=1}^{4} K_i Y_i \geq \sum_{j=1}^{8} D_j$ *was included in the master problem in order to guarantee that only solutions with sufficient capacity to meet the demand were produced.) The result was:*

```
Optimum of Master Problem

Optimal set of plants: 2 3 4
        with estimated cost [      ]
```

b. What is the value of the estimated cost of Y=(0,1,1,1) found by the Master problem ("blanked") above?

Next, the subproblem was solved, using the trial set of plants {2,3,4}, resulting in:

```
Subproblem Solution                     Generated support is αY+b, where
                                         α = 498 653 1364 639
Plants opened: # 2 3 4                   & b = ¯1096
                                        That is, v(Y) ≥ αY+b
Minimum transport cost = 1319
Fixed cost of plants = [      ]         This is support # 2
               Total = [      ]
```

c. What are the two values blanked above?

When the Master problem is solved once more, the result is:

```
Optimum of Master Problem

Optimal set of plants: 1 2 4

        with estimated cost [      ]
```

d. Compute the estimated cost found by the master problem.

The subproblem is again solved, this time with trial set {1,2,4}, yielding:

```
Subproblem Solution                     Generated support is αY+b, where
                                         α = 1144 668 89 [      ]
Plants opened: # 1 2 4                   & b = ¯1270
                                        That is, v(Y) ≥ αY+b
Minimum transport cost =  752
Fixed cost of plants   =  650           This is support # 3
                 Total = 1402
```

e. Using the information below from the subproblem solution, compute the blanked value of α_4 above.

Optimal Shipments

from \ to	1	2	3	4	5	6	7	8	9
1	6	0	0	0	5	8	0	0	0
2	0	1	2	0	0	0	3	0	9
4	0	0	0	5	0	0	0	10	2

Dual Variables

Supply constraints

i=	1	2	3	4
U[i]=	34	43	0	43

Demand constraints

j=	1	2	3	4	5	6	7	8
V[j]=	¯34	¯43	0	¯43	¯17	4	¯24	¯21

(Demand pt #9 is dummy demand for excess capacity.)

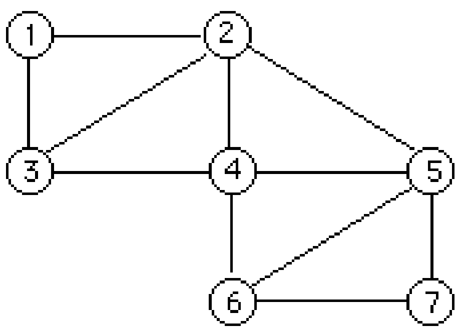
Reduced costs: COST - U°.+V

0	51	41	42	0	0	23	17
33	0	0	32	30	0	0	30
109	86	0	89	99	86	86	88
24	32	3	0	10	14	21	0

f. The solution of the transportation problem (the shipments) is degenerate. Based upon the information given, which variable(s) is/are basic but zero in the solution?

4. Integer Programming Model Formulation

a. Consider the street network below:



The postal service wishes to place the smallest number of mailboxes at intersections such that there is a mailbox at *one or both* ends of each of the eleven streets in the network. Define Y_i = 1 if a mailbox is placed at node i, and 0 otherwise. Formulate the problem as an integer LP.

b. For the same network above, formulate an integer LP (defining Y_i as before) to maximize the number of mailboxes such that *no* street has a mailbox at *both* ends.

c. Suppose that we consider each street segment (i.e., edge) to have length 1. We wish to place exactly two mailboxes at the nodes, and to assign each of the nodes to a mailbox so that the distance from the farthest node to its assigned mailbox is minimized. Define X_{ij} = 1 if node j is assigned to a mailbox at node i, 0 otherwise. Formulate the problem as an integer LP.

d. Which of the problems above (a, b, and/or c) are of the type known as a set covering problem? ___

e. Which of the problems above (a, b, and/or c) are of the type known as a p-center problem? ___

5. Traveling Salesman Problem. Five products are to be manufactured weekly on the same machine. The table below gives the cost of switching the machine from one product to another product. (Assume that this is also the cost of switching to the last product of the week to the first product to be scheduled the following week!)

from: \ to:	A	B	C	D	E
A	-	3	5	7	6
B	3	-	1	8	2
C	4	1	-	8	3
D	1	3	1	-	5
E	5	3	2	6	-

a. The nearest neighbor heuristic, starting with product A, yields the product sequence ___________ with cost ____.

After applying the "Hungarian Method" to the above matrix to solve the associated assignment problem (with large number, M, inserted along the diagonal), we have:

from: \ to:	A	B	C	D	E
A	M	0	2	0	2
B	2	M	0	3	0
C	3	0	M	3	1
D	0	2	0	M	3
E	3	3	0	0	M

b. What is the solution of this assignment problem?______________________________

c. What is its cost? _________

d. Is it a valid product sequence? _______ If not, why not? __________________________

e. If <u>not</u> a valid sequence, what bound (circle: upper / lower) on the optimal cost does this result provide? ________

f. If not a valid sequence, what single constraint might be added to the assignment problem to eliminate the solution which you have obtained (but not eliminate any valid sequence)?

g. If the assignment problem does not yield a valid sequence, how might we branch to create subproblems in a branch and bound method? (That is, specify the two direct descendents of the original problem.)

Lagrangian Relaxation: A small airline must schedule a departure from Cedar Rapids Airport to each of four cities: Des Moines, Minneapolis, Chicago, St. Louis. The available departure times are 1 pm, 2 pm, and 3 pm. The airline has only 2 departure lounges, and so at most two flights can be scheduled during a time slot. The airline estimates the following profits per flight (in hundreds of dollars) as a function of departure time:

	Departure time		
Destination	1:00	2:00	3:00
1. Des Moines	10	9	8

2. Minneapolis	11	9	9
3. Chicago	12	10	9
4. St. Louis	10	11	10

Define decision variables X_{ij} = 1 if flight to destination i is scheduled at time j,
and 0 otherwise (i=1,2,3,4; j=1,2,3).

a. Formulate the problem of maximizing profit as an integer LP.

b. Find (by inspection) a feasible solution to the problem. What is its profit? Is this an upper or lower bound on the optimal profit?

c. How would you apply Lagrangian relaxation to this formulation? (That is, select one or more constraints which you might relax.)

d. Assign a value of 5 to each Lagrange multiplier, and demonstrate how you would solve the Lagrangian relaxation.

e. What is the bound on the profit provided by this solution of the Lagrangian relaxation? Is it an upper or lower bound?

f. Is the solution found in (d) feasible? If not, demonstrate how you would adjust the multipliers so as to improve the bound found in (e) and/or improve feasibility.

g. Modify your formulation to handle the following conditions. (Define any additional decision variables which you might find necessary.)

(i) Flights to Chicago and St. Louis cannot depart during the same time slot.

(ii) The flight to Des Moines should depart before the flight to Chicago.

(iii) There is a cost of 1 for each time slot in which one or more departures are scheduled.