56:272 Integer Pgmg & Network Flows
Final Exam -- December 16, 1993
Instructor: D. L. Bricker

Where possible, answer the questions in the space provided; if additional space is necessary, use the back of the exam page or attach a page.

Do Part One (25 points), and <u>four</u> from Part Two (25 points each)

Score:	Part One:	I. _______	Multiple Choice
	Part Two:	II. _______	Location on network (median)
		III. _______	Location on network (center)
		IV. _______	Generalized assignment problem
		V. _______	Benders' decomposition
		VI. _______	Integer LP Model Formulation
		VII. _______	Lagrangian Relaxation
	TOTAL:	_______ (of 125 possible)	

PART ONE

I. ***True/False***

____ 1. The quadratic assignment problem has terms involving $(X_{ij})^2$ in the objective function.

____ 2. Balas' implicit enumeration algorithm does not require multiplication or division.

____ 3. The center location problem is to choose a facility location to minimize the distance to the farthest "customer".

____ 4. The optimal objective function value for the median problem in a network is less than the optimal objective function value for the center problem.

____ 5. A heuristic algorithm for a combinatorial problem does not guarantee an optimal solution.

____ 6. If the LP relaxation of an integer LP is infeasible, then the integer LP is infeasible.

____ 7. If the LP relaxation of an integer LP has an integer solution, that solution is optimal for the integer LP.

____ 8. For a minimization ILP problem, the LP relaxation provides a lower bound on the optimal value of the ILP.

____ 9. For a minimization ILP problem, the LP relaxation cannot provide a better (i.e., larger) lower bound than a Lagrangian relaxation (for arbitrary values of the Lagrangian multipliers).

____ 10. For a minimization problem, each subproblem solution in Benders' decomposition algorithm is as low as or lower than previous subproblem solutions.

____ 11. For a minimization problem, the optimal value of each master problem solution (when the master problem is optimized) is greater than or equal to the previous master problem solution.

____ 12. For a minimization problem, the optimal value of each master problem solution (when the master problem is optimized) is greater than or equal to the optimal value of the subproblem which follows.

____ 13. For Lagrangian relaxation applied to the set covering problem, if all of the linear constraints are relaxed, the greatest lower bound which can be achieved is the same as the LP relaxation.

____ 14. For Lagrangian relaxation applied to a minimization problem, the Lagrangian dual problem is a maximization problem.

____ 15. In Gomory's fractional cutting plane algorithm, the number of rows in the LP being solved increases at every iteration.

____ 16. In Gomory's fractional cutting plane algorithm applied to a minimization problem, the objective function decreases (or stays the same) at every iteration.

____ 17. In Gomory's fractional cutting plane algorithm, after the first LP relaxation is solved, any later pivoting is done only on negative elements in the tableau.

____ 18. In the generalized assignment problem, there is a setup cost for each machine which is used.

____ 19. The quadratic assignment problem is frequently used to assign facilities to locations where there is a specified flow between certain pairs of facilities and a cost of this flow which depends upon distances between the facilities.

____ 20. The constraints of the quadratic assignment problem and the linear (i.e., the "classical") assignment problem are identical.

____ 21. The linear (i.e., the "classical") assignment problem is a special case of the generalized assignment problem.

____ 22. Benders' decomposition method partitions the decision variables into two sets, whereas Lagrangian relaxation partitions the constraints into two sets.

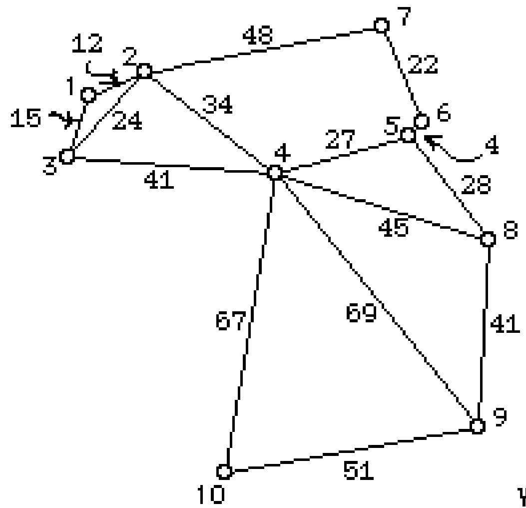
____ 23. The optimal values of the decision variables in the LP relaxation of the "Simple Plant Location Problem" are integer.

____ 24. In both the "simple plant location problem" and the "k-median problem", it is assumed that a facility can supply an unlimited amount of goods to the customers.

____ 25. The optimal value of the Lagrangian dual problem resulting from a Lagrangian relaxation of an ILP is equal to the optimal value of the ILP.

PART TWO

II. ***The Median Plant Location Problem:*** Consider the network below, where the "weight" Wt[i] is the quantity to be delivered to city i:



Weighted Shortest Path Lengths

from \ to	1	2	3	4	5	6	7	8	9	10
1	0	12	45	138	146	308	240	364	115	339
2	12	0	72	102	122	260	192	316	103	303
3	15	24	0	123	136	288	288	344	110	324
4	46	34	123	0	54	124	212	180	69	201
5	73	61	204	81	0	16	104	112	69	282
6	77	65	216	93	8	0	88	128	73	294
7	60	48	216	159	52	88	0	216	95	360
8	91	79	258	135	56	128	216	0	41	276
9	115	103	330	207	138	292	380	164	0	153
10	113	101	324	201	188	392	480	368	51	0
Wt[i]	1	1	3	3	2	4	4	4	1	3

a. Formulate the 3-median problem as a binary integer LP. (Be sure to state the definition of your decision variables.)

b. Can your ILP model in (a) be solved by Balas' Algorithm? *Circle:* Yes No
If not, explain why:

The addition/substitution heuristic was applied to try to find the 3-median set, giving the output below:

1-Median

5
Cost = 1002

2-Median

Trial additions:

Add	1	2	3	4	6	7	8	9	10
cost	721	748	703	705	970	872	856	804	702

Addition result: Locations A,B

Cost: C

Substitution Step

Cost	Locations	Cost	Locations
1304	1 10	726	6 10
721	5 1	970	5 6
1127	2 10	890	7 10
748	5 2	872	5 7
1269	3 10	1004	8 10
703	5 3	856	5 8
824	4 10	1713	9 10
705	5 4	804	5 9

No substitution can be made

3-Median

Trial additions:

Add	1	2	3	4	6	7	8	9
cost	421	448	403	486	670	572	580	651

Addition result: Locations D,E,F

Cost: G

Substitution Step

Cost	Locations	Cost	Locations
1194	1 10 3	671	5 6 3
676	5 1 3	670	5 10 6
421	5 10 1	569	7 10 3
1055	2 10 3	599	5 7 3
676	5 2 3	572	5 10 7
448	5 10 2	603	8 10 3
660	4 10 3	557	5 8 3
541	5 4 3	580	5 10 8
486	5 10 4	1038	9 10 3
407	6 10 3	505	5 9 3

c. Four values are blanked in the output of the addition/substitution heuristic. What are these values?

A,B ______________ (the result of the addition step for the 2-median)

C ________________ (the cost of the set {A,B})

D,E,F _____________ (result of the addition step for the 3-median)

G. ________________ (the cost of the set {D,E,F})

d. What is the 3-median set found by the substitution step of this algorithm?
{ ___ , ___ , ___ }

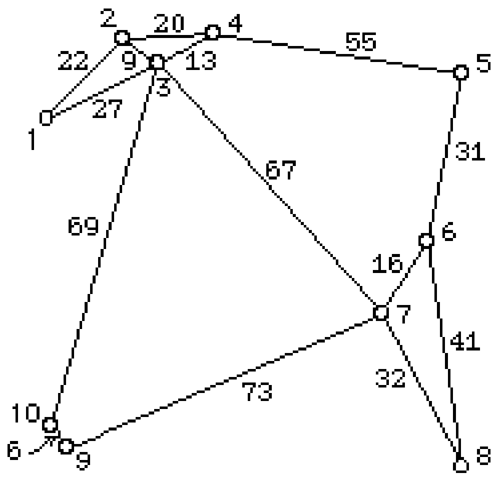
e. What is the cost of the set given in (d)? ____________

f. What cities are served by each facility?

Facility	Cities served
____	_________________
____	_________________
____	_________________

g. Suppose that the facilities are not required to be at nodes of the network, but could also be located anywhere on an edge. How would you solve the problem in this case?

III. ***Location of Center of Network*** Consider the undirected network below:



Shortest Path Lengths

from \ to	1	2	3	4	5	6	7	8	9	10
1	0	22	27	40	95	110	94	126	102	96
2	22	0	9	20	75	92	76	108	84	78
3	27	9	0	13	68	83	67	99	75	69
4	40	20	13	0	55	86	80	112	88	82
5	95	75	68	55	0	31	47	72	120	126
6	110	92	83	86	31	0	16	41	89	95
7	94	76	67	80	47	16	0	32	73	79
8	126	108	99	112	72	41	32	0	105	111
9	102	84	75	88	120	89	73	105	0	6
10	96	78	69	82	126	95	79	111	6	0

a. Find the **vertex** center for this network (with unweighted distances): City # ____

b. Give a mathematical expression for the objective function which is being minimized in the problem of finding the **absolute** center of the network in terms of d(x,k), the length of the shortest path from x to city k.

c. Below is some output displaying a lower bound which may be computed for the center objective function on each edge. What is the missing value? __________

Lower Bounds on Local Edge-Centers

i	j	LB
3	7	63
7	9	70.5
3	10	78
4	5	91.5
6	7	94
7	8	94
6	8	97.5
1	3	99
2	3	99
3	4	99
2	4	$\boxed{}$
5	6	102.5
1	2	106
9	10	120

d. How many edges (i,j) cannot be eliminated from consideration when searching for the absolute center? ____ (Circle them in the table in (c).)

e. Below is information about the center objective function on the edge (3,10). What are the three missing values?

Minimax Objective on Edge (3,10)

Monotonically increasing distance functions:

d(x,k) where

k=	1	2	3	4
d(i,k)=	27	9	0	13
d(j,k)=	96	78	69	$\boxed{}$

Monotonically decreasing distance functions:

d(x,k) where

k=	9	10
d(i,k)=	75	69
d(j,k)=	$\boxed{}$	0

Distance functions which increase to a p
at a point Δ units from i, then decre

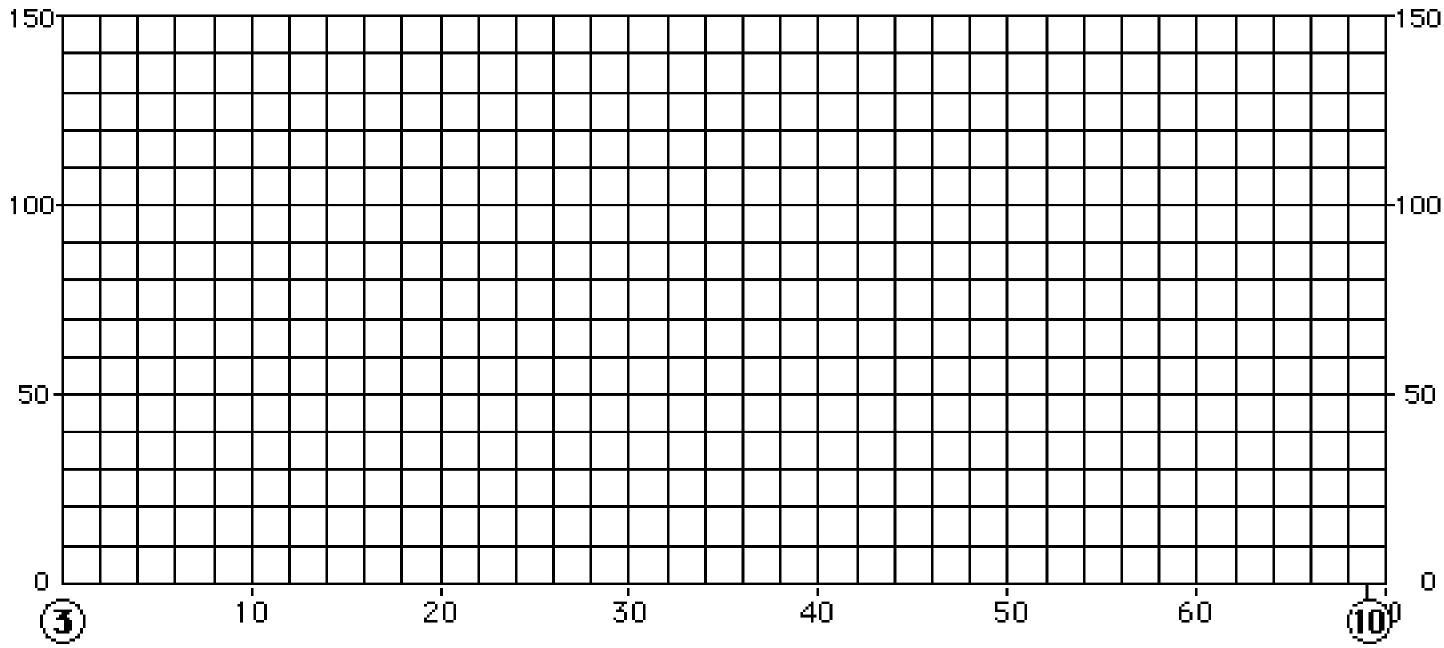
d(x,k) where

k=	5	6	7
d(i,k)=	68	83	67
d(j,k)=	126	95	79
Δ=	63.5	40.5	40.5

f. Sketch the center objective function on the edge (3,10). How far from city #3 is the edge center of (3,10)? ______

g. What is the objective function at the edge center of the edge (3,10)? _________

h. Given the information which you now have, could the edge center of (3,10) possibly be the absolute center of the network? *Circle:* Yes No



IV. Generalized Assignment Problem: Consider the problem of assigning 6 jobs to 3 machines (each with limited capacity):

Resources Used

Machine i	Job 1	Job 2	Job 3	Job 4	Job 5	Job 6	Available b
1	18	22	24	21	24	16	20
2	23	21	18	15	24	6	27
3	15	21	14	15	13	14	38

Costs

Machine i	Job 1	Job 2	Job 3	Job 4	Job 5	Job 6
1	10	18	11	17	24	13
2	11	20	25	24	18	13
3	23	11	17	18	18	19

a. Formulate this problem as a binary integer programming problem.

b. Suppose that the machine capacity constraints are relaxed, using the Lagrangian relaxation method. The first 2 iterations of the subgradient optimization method to maximize the lower bound appears below, where the optimal value was estimated to be 120, and a stepsize parameter was assigned the value 0.75.

```
Lambda = 0.75
Upper bound Z* = 120
```

Iteration # 1

```
Multiplier vector U = 0 0 0
Objective function of relaxation:
```

machine \ job	1	2	3	4	5	6
1	10	18	11	17	24	13
2	11	20	25	24	18	13
3	23	11	17	18	18	19

```
Dual value is [ A ]
Variables selected from GUB sets are:
    1 3 1 1 2 1
Resources used are: 79 24 21, (Available: 20 27 38)
Subgradient of Dual Objective is [ B | C | D ]
Stepsize is 0.00861821
```

Iteration # 2

```
Multiplier vector U = 0.508475 0 0
Objective function of relaxation:
```

machine \ job	1	2	3	4	5	6
1	19.15	29.18	23.20	27.67	36.20	21.13
2	11	20	25	24	18	13
3	23	11	17	18	18	19

```
Dual value is 77.8305
Variables selected from GUB sets are:
    [ E | F | G | H | I | J ]
Resources used are: 0 53 50, (Available: 20 27 38)
Subgradient of Dual Objective is ¯20 26 12
Stepsize is [ K ]
```

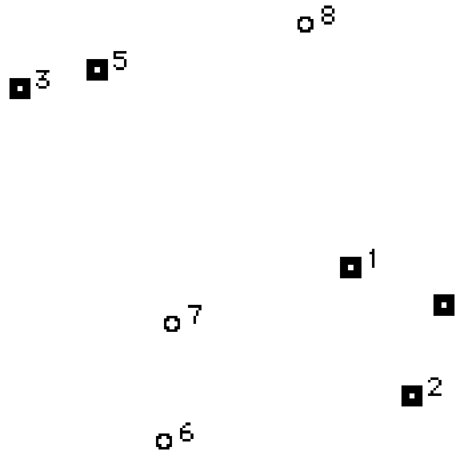
c. Several values have been omitted from the output. Compute their values:

A _____ B _____ C _____ D ______ E _____
F _____ G _____ H _____ I ______ J _____
K _____

d. Does this Lagrangian relaxation possess the "integrality property"? *Circle:* Yes No
Why or why not?

e. What does your answer in (d) imply about the strength of the lower bound which can be obtained from this relaxation, compared to that of the LP relaxation?

V. Benders' Decomposition Algorithm for Plant Location: Consider the following randomly-generated problem in which demand in 8 cities is to be satisfied by building plants in one or more of four of the cities:



Costs, Supplies, Demands

i \ j	1	2	3	4	5	6	7	8	K	F
1	0	25	67	18	57	45	34	44	10	5000
2	25	0	89	17	81	45	45	69	17	3000
3	67	89	0	85	14	68	50	52	10	7000
4	18	17	85	0	75	55	49	56	14	4000
5	57	81	14	75	0	67	47	38	18	1000
Demand:	3	5	7	3	8	4	2	9	69	

Number of sources = M = 5
Number of destinations = N = 8
Total demand: 41

a. State the mixed-integer programming formulation of the problem. How many continuous variables (X) and how many binary (zero-one) variables (Y) are required?

b. Give the expression for the optimal value as a function of Y, i.e. v(Y), expressed in terms of the variables X.

A trial solution was evaluated, in which plants 1, 2, 3, & 4 are to be open. The result was:

Subproblem Solution

Plants opened: # 1 2 3 4

Minimum transport cost = 1077
Fixed cost of plants = 19000
Total = 20077

Generated support is αY+b, where
α = 5570 4173 7140 4966 1000
& b = ¯1772

This is support # 1

Optimal Shipments

from \ to	1	2	3	4	5	6	7	8	9
1	3	0	0	0	5	0	0	2	0
2	0	5	0	0	0	4	2	0	6
3	0	0	7	0	3	0	0	0	0
4	0	0	0	3	0	0	0	7	4

(Demand pt #9 is dummy demand for excess capacity.)

c. Is the optimal solution of this subproblem degenerate? *Circle:* Yes No
Why or why not?

Next a suboptimal solution of the Master Problem is found:

```
Master Problem

(suboptimized, i.e., a solution
   Y such that v(Y) < incumbent.)
Trial set of plants : <empty>
with estimated cost [          ]
Current status vectors for Balas'
 additive algorithm:
     j:      ¯1  ¯2  ¯3  ¯4  ¯5
underline:   0   0   0   0   0
```

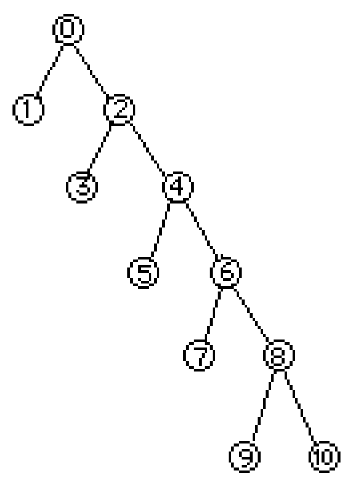
d. What is the value $\underline{v}_1(0,0,0,0,0)$ of the master problem objective which is blanked out above?

Using this solution of the master problem (which was sub-optimized), the subproblem, i.e. transportation problem, was next solved:

```
Subproblem Solution

Plants opened: # <empty>

Minimum transport cost = 410000
Fixed cost of plants = 0
       Total = 410000
Generated support is αY+b, where
 α = ¯95000 ¯167000 ¯93000 ¯136000 ¯179000
 & b = 410000
This is support # 2
```

Next the master problem is sub-optimized again:

```
Master Problem

(suboptimized, i.e., a solution Y such
   that v(Y) < incumbent.)
Trial set of plants: 3 4 5
with estimated cost [          ]
Current status vectors for Balas'
  additive algorithm:
     j:      ¯1  ¯2  3   5   4
underline:   0   0   1   0   0
```



e. What is the value blanked out in the master problem solution above?

f. Suppose that node #10 on the implicit enumeration tree above represents the master problem solution. Which nodes have already been fathomed? ________________

g. Which variables have been fixed at node #6? ______________

h. After node #10 is fathomed, which node is considered next in the implicit enumeration?
____

Next the subproblem was solved, using the set of plants {3, 4, 5}:

Subproblem Solution

Plants opened: # 3 4 5

Minimum transport cost = 811
Fixed cost of plants = 12000
Total = 12811

Generated support is αY+b, where
α = 5000 3000 7310 4252 1504
& b = ¯255

This is support # 3

i. Suppose that we wish to estimate the cost of the proposal to open plants #1, 3, 4, &5.

Current List of Supports of v(Y)

Open plants: 1 3 4 5

Current approximation of v(Y) is
Maximum { α[i]Y + b[i] }
where α & b are:

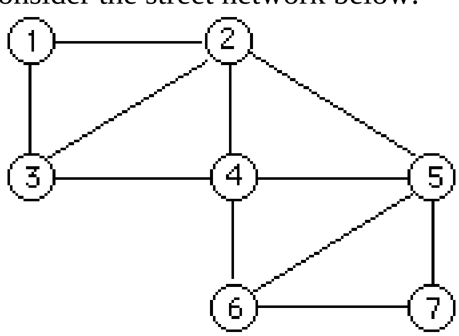
α_1	α_2	α_3	α_4	α_5	b
5570	4173	7140	4966	1000	¯1772
¯95000	¯167000	¯93000	¯136000	¯179000	410000
5000	3000	7310	4252	1504	¯255

support k	$\sum_{i=1}^{5} \alpha_i^k Y_i + \beta^i$
1	16904
2	¯93000
3	17811

Does this give us an over- or under-estimate of the cost? Could the set of plants {1,3,4,5} possibly be optimal? Why or why not?

VI. Integer Programming Models

1. Consider the street network below:



The postal service wishes to place the smallest number of mailboxes at intersections such that there is a mailbox at one or both ends of each of the eleven streets in the network. Define $Y_i = 1$ if a mailbox is placed at node i, and 0 otherwise. Formulate the problem as an integer LP.

2. For the same network above, formulate an integer LP (defining Y_i as before) to maximize the number of mailboxes such that no street has a mailbox at both ends.

3. Suppose that each street has length 1. We wish to place exactly two mailboxes at the nodes, and to assign each of the nodes to a mailbox so that the distance from the farthest node to its assigned mailbox is minimized. Define $X_{ij} = 1$ if node j is assigned to a mailbox at node i, 0 otherwise. Formulate the problem as an integer LP.

4. Which of the problems above (1, 2, and/or 3) are of the type known as set covering problems?

VII. Lagrangian Relaxation: A small airline must schedule a departure from Cedar Rapids Airport to each of four cities: Des Moines, Minneapolis, Chicago, St. Louis. The available departure times are 1 pm, 2 pm, and 3 pm. The airline has only 2 departure lounges, and so at most two flights can be scheduled during a time slot. The airline estimates the following profits per flight (in hundreds of dollars) as a function of departure time:

	Departure time		
Destination	1:00	2:00	3:00
1. Des Moines	10	9	8
2. Minneapolis	11	9	9
3. Chicago	12	10	9
4. St. Louis	10	11	10

Define decision variables $X_{ij} = 1$ if flight to destination i is scheduled at time j, and 0 otherwise (i=1,2,3,4; j=1,2,3).

1. Formulate the problem of maximizing profit as an integer LP.

2. Find a feasible solution to the problem. What is its profit? Is this an upper or lower bound on the optimal profit?

3. How would you apply Lagrangian relaxation to this formulation? (That is, select one or more constraints which you might relax.)

4. Assign a value of 5 to each Lagrange multiplier, and demonstrate how you would solve the Lagrangian relaxation.

5. What is the bound on the profit provided by this solution of the Lagrangian relaxation? Is it an upper or lower bound?

6. Is the solution found in (4) feasible? If not, demonstrate how you would adjust the multipliers so as to improve the bound found in (5) and/or improve feasibility.

7. Modify your formulation to handle the following conditions. (Define any additional decision variables which you might find necessary.)

   a. Flights to Chicago and St. Louis cannot depart during the same time slot.

   b. The flight to Des Moines should depart before the flight to Chicago.

   c. There is a cost of 1 for each time slot in which one or more departures are scheduled.