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@ 56:272 Integer Programming & Network Flows @

@@@ Exam #3 - December 13, 1994 @@@

Answer all five problems.

	possible	score	
1. ILP Models	15	_____	
2. Balas' Additive Algorithm	15	_____	
3. Benders' Decomposition	30	_____	
4. Lagrangian Relaxation	25	_____	Total
5. Cutting Planes	15	_____	_____

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1. Integer LP Model Formulation The board of directors of a large manufacturing firm is considering the set of investments shown below: Let R_i be the total revenue from investment i and C_i the cost (in $millions) to make investment i. The board wishes to maximize total revenue and invest no more than a total of 50 million dollars.

Investment i	Revenue R_i	Cost C_i	Condition
1	1	5	None
2	2	8	Only if #1
3	3	12	Only if #2
4	4	18	Must if #1 and #2
5	5	24	Not if #1 or #2
6	6	27	Not if both #2 and #3
7	7	30	Only if #2 and not #3

Define X_i = 1 if investment i is selected, else 0. Formulate this problem as an integer LP.

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2. Balas' Additive Algorithm (Implicit Enumeration) Consider the problem

$$
\begin{array}{lrrrrrcl}
\text{Maximize} & -7X_1 & +6X_2 & +3X_3 & -5X_4 & -6X_5 & & \\
\text{subject to} & -2X_1 & -X_2 & +2X_3 & +3X_4 & -3X_5 & \le & 1 \\
 & & X_2 & -2X_3 & -4X_4 & -2X_5 & \le & -1 \\
 & 2X_1 & & +4X_3 & +3X_4 & & \le & 4 \\
 & \multicolumn{7}{l}{X_i = 0 \text{ or } 1,\ i=1,2,\ldots 5}
\end{array}
$$

a. Convert the problem to the standard form, i.e.,

$$
\begin{array}{ll}
\text{Minimize} & \Sigma\, c_i x_i \\
\text{subject to} & Ax \le b \\
 & X_i = 0 \text{ or } 1
\end{array}
$$

with $c_i \ge 0$ for all i

b. Suppose you are now at the node represented by J = {<u>5</u>, -2, <u>3</u> } (where indices are of the variables in the standard form you found in (a).) Draw the corresponding tree diagram, and indicate this current node. Which are the free variables? Which are variables fixed at 0? Which are variables fixed at 1? What do the underlines indicate?

c. Try to fathom the current node above by the appropriate fathoming tests of Balas' algorithm. Can it be fathomed?

Name or Initials ____________________

d. Regardless of your answer in (c), suppose that the node is fathomed. Which node (i.e., which vector J) is to be considered next?

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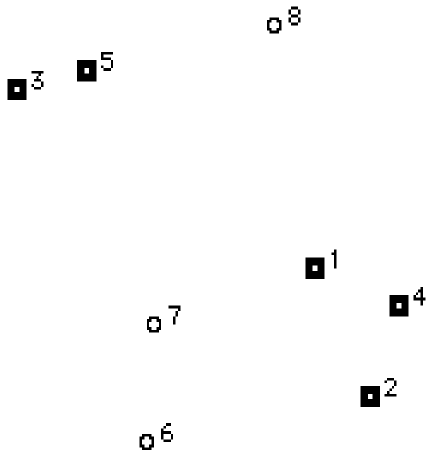
3. Benders' Decomposition Algorithm for Plant Location: Consider the following randomly-generated problem in which demand in 8 cities is to be satisfied by building plants in one or more of four of the cities:



Costs, Supplies, Demands

i \ j	1	2	3	4	5	6	7	8	K	F
1	0	25	67	18	57	45	34	44	10	5000
2	25	0	89	17	81	45	45	69	17	3000
3	67	89	0	85	14	68	50	52	10	7000
4	18	17	85	0	75	55	49	56	14	4000
5	57	81	14	75	0	67	47	38	18	1000
Demand:	3	5	7	3	8	4	2	9	69	

Number of sources = M = 5
Number of destinations = N = 8
Total demand: 41

a. State the mixed-integer programming formulation of the problem. How many continuous variables (X) and how many binary (zero-one) variables (Y) are required?

b. Give the expression for the optimal value as a function of Y, i.e. v(Y), expressed in terms of the variables X.

A trial solution was evaluated, in which plants 1, 2, 3, & 4 are to be open. The result was:

Subproblem Solution

Plants opened: # 1 2 3 4

Minimum transport cost = 1077
Fixed cost of plants = 19000
Total = 20077

Generated support is αY+b, where

α = 5570 4173 7140 4966 1000

& b = ¯1772

This is support # 1

Optimal Shipments

from \ to	1	2	3	4	5	6	7	8	9
1	3	0	0	0	5	0	0	2	0
2	0	5	0	0	0	4	2	0	6
3	0	0	7	0	3	0	0	0	0
4	0	0	0	3	0	0	0	7	4

(Demand pt #9 is dummy demand for excess capacity.)

c. Is the optimal solution of this subproblem degenerate? *Circle:* Yes No
Why or why not?

Next a suboptimal solution of the Master Problem is found:

```
Master Problem

(suboptimized, i.e., a solution
   Y such that v(Y) < incumbent.)
Trial set of plants : <empty>
with estimated cost [          ]
Current status vectors for Balas'
 additive algorithm:
    j:     ¯1 ¯2 ¯3 ¯4 ¯5
underline:  0  0  0  0  0
```

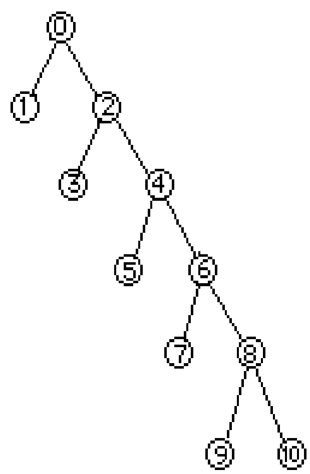
d. What is the value $\underline{v}_1(0,0,0,0,0)$ of the master problem objective which is blanked out above?

Using this solution of the master problem (which was sub-optimized), the subproblem, i.e. transportation problem, was next solved:

```
Subproblem Solution

Plants opened: # <empty>

Minimum transport cost = 410000
Fixed cost of plants = 0
       Total = 410000
Generated support is αY+b, where
 α = ¯95000 ¯167000 ¯93000 ¯136000 ¯179000
 & b = 410000
This is support # 2
```

(The transportation problem is infeasible; the given cost is actually the cost of the artificial variables, which have high cost coefficiencts.) Next the master problem is sub-optimized again:

```
Master Problem

(suboptimized, i.e., a solution Y such
   that v(Y) < incumbent.)
Trial set of plants: 3 4 5
with estimated cost [          ]
Current status vectors for Balas'
  additive algorithm:
    j:     ¯1 ¯2  3  5  4
underline:  0  0  1  0  0
```



e. What is the value blanked out in the master problem solution above?

f. Suppose that node #10 on the implicit enumeration tree above represents the master problem solution. Which nodes have already been fathomed? __________________

g. Which variables were fixed at node #8? _______________

Name or Initials ____________________

h. After node #10 is fathomed, which node is considered next in the implicit enumeration? _______

Next the subproblem was solved, using the set of plants {3, 4, 5}:

Subproblem Solution

Plants opened: # 3 4 5

Minimum transport cost = 811
Fixed cost of plants = 12000
Total = 12811

Generated support is αY+b, where
α = 5000 3000 7310 4252 1504
& b = ¯255

This is support # 3

Suppose that, using the three current supports which have been generated, we wish to estimate the cost of the proposal to open plants #1, 3, 4, &5. The value of each support at Y=(1,0,1,1,1) is given in the column on the right below:

Current List of Supports of v(Y)

Open plants: 1 3 4 5

Current approximation of v(Y) is
Maximum { α[i]Y + b[i] }
where α & b are:

α_1	α_2	α_3	α_4	α_5	b
5570	4173	7140	4966	1000	¯1772
¯95000	¯167000	¯93000	¯136000	¯179000	410000
5000	3000	7310	4252	1504	¯255

support k	$\sum_{i=1}^{5} \alpha_i^k Y_i + \beta^i$
1	16904
2	¯93000
3	17811

i. What is the value of $\underline{v}_3(1,0,1,1,1)$? _____________

j. Does this give us an over- or under-estimate of the cost of opening plants #1, 3, 4, & 5?

k. Could the set of plants {1,3,4,5} possibly be optimal? Why or why not?

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4. Lagrangian Relaxation: Consider the following plant location problem:

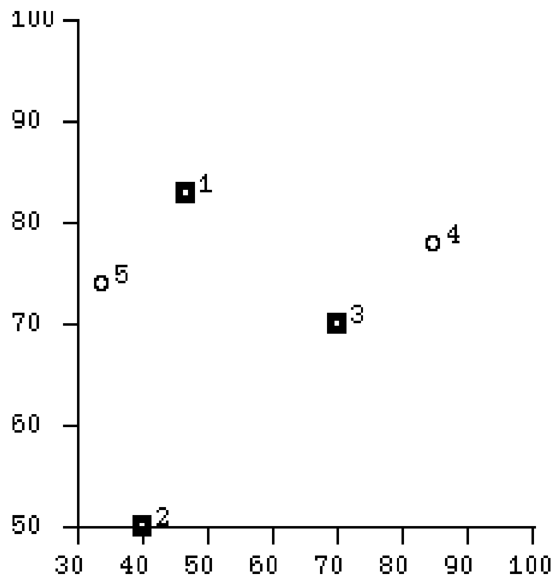
Name or Initials ____________________

Number of sources = M = 3
Number of destinations = N = 5
Total demand: 21

Costs, Supplies, Demands

i/j	1	2	3	4	5	K	F
1	0	30	30	40	20	11	5000
2	30	0	40	50	30	8	2000
3	30	40	0	20	40	9	3000
Demand:	6	2	9	2	2	28	0

K = capacity,
F = fixed cost



a. Formulate the problem as a mixed-integer LP problem:

b. Apply Lagrangian relaxation to relax the supply constraints of your formulation. Write the Lagrangian subproblem to be optimized for fixed values of the Lagrangian multipliers.

c. Let the Lagrange multipliers for the three supply constraints be equal to 100, 50, and 0, respectively. Illustrate how the Lagrangian subproblem is easily solved.

d. What is the optimal objective value of the Lagrangian subproblem in (c) above? Is it a lower or upper bound on the optimum of the original problem?

e. For each of the Lagrange multipliers, specify whether it should be adjusted upward or downward in order to improve the bound.

f. Does this Lagrangian subproblem have the "Integrality Property"? What does your answer imply about the quality of the bound compared to that of LP relaxation?

Name or Initials ____________________

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5. Cutting Plane Algorithm. Consider the problem

$$\begin{array}{ll} \text{Minimize} & -8X_1 - 5X_2 \\ \text{subject to} & X_1 + X_2 \leq 6 \\ & 9X_1 + 5X_2 \leq 45 \\ & X_1,\ X_2 \geq 0,\ \&\ \text{integer} \end{array}$$

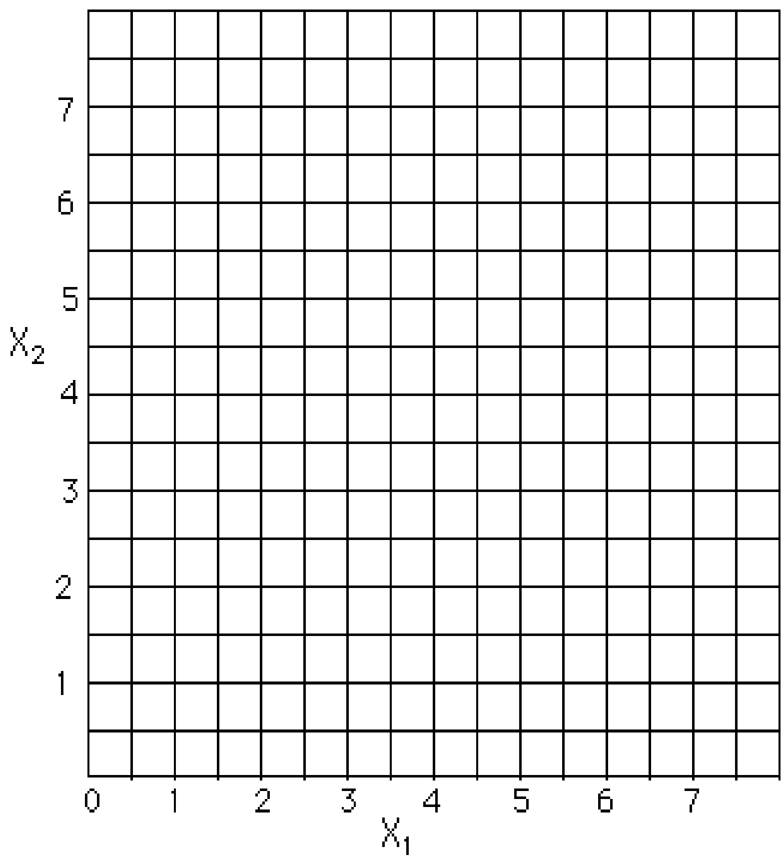
After adding slack & surplus variables X_3 & X_4, respectively, and solving the LP relaxation, we get the optimal tableau:

X_1	X_2	X_3	X_4	rhs	
0	0	1.25	0.75	41.25	(min)
1	0	2.25	-0.25	2.25	
0	1	-1.25	0.25	3.75	

a. What two constraints may be added to the problem to exclude this extreme point of the feasible region of the LP relaxation without excluding any integer feasible solutions?

b. Choose one of these constraints which you found in (a), and express it in terms of the original variables X_1 and X_2.

c. Graph, in (X_1, X_2)-space, the original constraints, the optimum of the LP relaxation, and the new constraint which you chose in (b). Shade the feasible region after adding this constraint.



c. Add the new constraint in (b) to the tableau, and indicate where the next pivot should be:

	X_1	X_2	X_3	X_4	____	rhs
	0	0	1.25	0.75		41.25 (min)
	1	0	2.25	-0.25		2.25
	0	1	-1.25	0.25		3.75
added row:						