



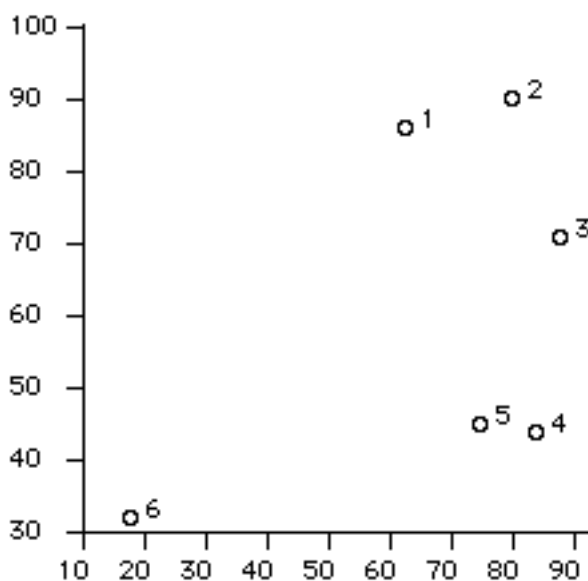
56:272 Integer Prgmg & Network Flows
Take-Home Final Exam -- Fall 1991
Due noon, Friday, December 20, 1991

Instructor: D. L. Bricker



Where possible, answer the questions in the space provided; if additional space is necessary, use the back of the exam page or attach a page. You may consult any books, notes, etc., but not other persons!

(1.) Traveling Salesman Problem Consider the 5 cities below, which are each to be visited by a vehicle which will begin and end its route at city #1. (*Warning: the horizontal & vertical axes are scaled differently, so do not judge distances visually!*)

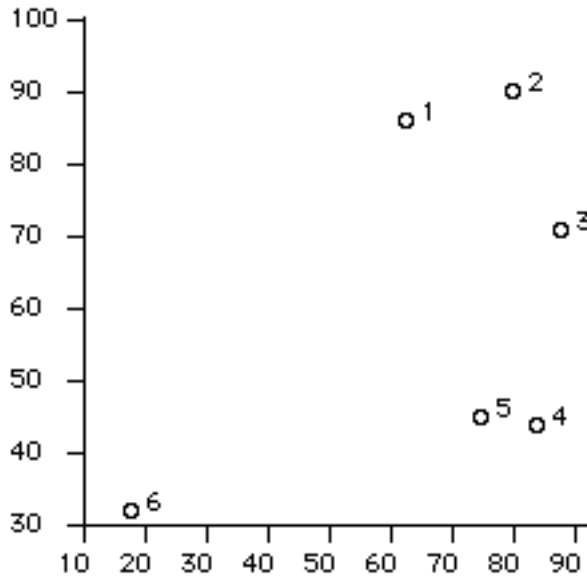


i	1	2	3	4	5	6
X[i]	63	80	88	84	75	18
Y[i]	86	90	71	44	45	32

Distances

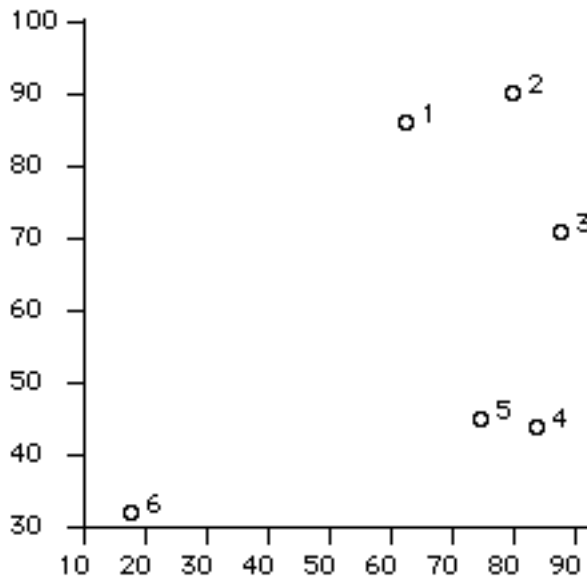
		to					
		1	2	3	4	5	6
1		0	17	29	47	43	70
f	2	17	0	21	46	45	85
r	3	29	21	0	27	29	80
o	4	47	46	27	0	9	67
m	5	43	45	29	9	0	58
	6	70	85	80	67	58	0

- a. Apply the nearest neighbor heuristic (starting at node 1) to the problem. What is the length of the tour?
- b. Find the minimum spanning one-tree of the nodes above (letting node #1 be the "root" node). Is it a tour?
- c. Assign vertex penalties and perform one iteration of the vertex penalty method, using a "unit penalty" of 10. Does it result in a tour? (Indicate the result below.)



d. Explain how the vertex penalty method may be interpreted as a Lagrangian relaxation method for the traveling salesman problem. What constraints are being "relaxed"? What is the objective function of the Lagrangian relaxation?

e. Perform a second iteration of the vertex penalty method. Is the result a tour?



f. Based upon your answers above, state an **upper** and a **lower** bound on the length of the optimal tour.

g. Suppose that the Hungarian algorithm were applied to the distance matrix of a TSP. Does the solution always satisfy the constraint relaxed in (d)?

(2.) Assignment Problem: Consider the problem of assigning 5 jobs to 5 machines, with the following cost matrix (where the element in row i & column j is the cost of assigning job i to machine j):

1	5	3	4	10
3	8	5	1	8
10	9	5	9	3
9	5	8	2	10
2	10	1	6	7

- Formulate this problem as a binary integer LP.
- If you apply the simplex algorithm to the LP relaxation of this problem, are you guaranteed of obtaining an integer solution? Why?
- How many basic variables does the LP relaxation of this problem have? How many are positive at the optimum? Such a basic solution of an LP has the property called _____.
- How do the generalized assignment problem and the quadratic assignment problem differ from the assignment problem such as was solved above? (How do the models differ mathematically? Describe differences in the applications which the three formulations might model.)

(3.) GOMORY'S FRACTIONAL CUTTING-PLANE ALGORITHM

Consider the problem:

$$\begin{array}{ll}
 \text{Maximize} & 3 X_1 - X_2 \\
 \text{subject to} & 3 X_1 - 2 X_2 \quad 3 \\
 & 5 X_1 + 4 X_2 \quad 10 \\
 & 2 X_1 + X_2 \quad 5 \\
 & X_1, X_2 \geq 0 \text{ \& integer}
 \end{array}$$

After adding slack & surplus variables X_3 , X_4 and X_5 , and solving the LP relaxation, we get the optimal tableau:

Current LP Tableau

z	1	2	3	4	5	B
1	0	0	-0.714	0	-0.429	-4.29
0	1	0	0.143	0	0.286	1.86
0	0	1	-0.286	0	0.429	1.29
0	0	0	-0.429	1	3.14	4.43

Variables:
 (Negative of) objective function value: z
 Original structural variables: 1 2
 Original slack/surplus variables: 3 4 5

- a. A constraint was added to the problem to exclude this optimal LP solution without excluding any integer feasible solution, using row #4 as the "source row":

Source row is # 4

i:	3	4	5	6	rhs
Source row:	-0.429	1	3.14	0	4.43
Cut:	a	0	-0.143	1	-0.429

(X[6] (= slack variable for new cut)
is basic but < 0)

What is the value which is blocked out above? **a** = _____

- b. The constraint in (a) may be expressed in terms of the original variables X_1 & X_2 :

The cut which is added is
 (in terms of original variables):

$$\frac{1}{2} X_1 - X_2 \leq \mathbf{b}$$

i.e., $2X_1 - X_2 \leq \mathbf{b}$ = _____

(4.) Generalized Assignment Problem: Consider the problem of assigning 4 jobs to 3 machines (each with limited capacity):

Costs					Resources Used					
Machine	Jobs				Machine	Jobs				Available
i	1	2	3	4	i	1	2	3	4	b
1	11	14	13	21	1	5	11	25	11	23
2	12	23	17	20	2	14	5	13	16	27
3	21	14	12	14	3	13	18	10	14	17

- a. Formulate this problem as a binary integer programming problem.

b. Suppose that the capacity constraints are relaxed, using the Lagrangian relaxation method. The first 3 iterations of the subgradient optimization method to maximize the lower bound appears below, where the value of a feasible solution was found to be 56, and a stepsize parameter was assigned the value 0.75.

Lambda = 0.75
Upper bound Z* = 56

Iteration # 1

Multiplier vector U = 0 0 0
Objective function of relaxation:

		to			
		1	2	3	4
f					
r	1	11	14	13	21
o	2	12	23	17	20
m	3	21	14	12	14

Dual value is 51
Variables selected from GUB sets are:
1 1 3 3
Resources used are: 16 0 24,
(Available: 23 27 17)
Subgradient of Dual Objective is -7 -27 7
Stepsize is 0.0765306

Iteration # 2

Multiplier vector U = 0 0 0.535714
Objective function of relaxation:

		to			
		1	2	3	4
f					
r	1	11	14	13	21
o	2	12	a	17	20
m	3	27.9643	23.6429	17.3571	b

Dual value is **c**
Variables selected from GUB sets are:
1 1 1 2
Resources used are: 41 16 0,
(Available: 23 27 17)
Subgradient of Dual Objective is 18 -11 -17
Stepsize is 0.00869553

Iteration # 3

Multiplier vector U = 0.156519 0 **d**
Objective function of relaxation:

		to			
		1	2	3	4
f					
r	1	11.7826	15.7217	16.913	22.7217
o	2	12	23	17	20
m	3	26.0426	20.982	15.8789	19.4305

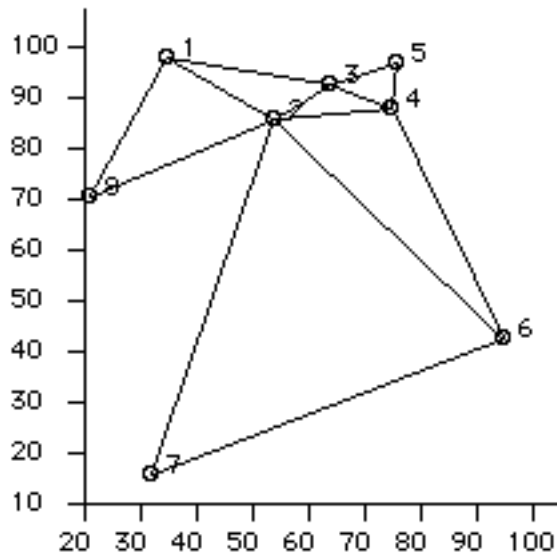
Dual value is 52.6196
Variables selected from GUB sets are:
1 1 3 3
Resources used are: 16 0 24,
(Available: 23 27 17)
Subgradient of Dual Objective is
e **f** **g**
Stepsize is 0.0223599

- c. Several values have been omitted from the output. Compute their values:
- a. _____ (cost coefficient of the Lagrangian relaxation)
 - b. _____ (cost coefficient of the Lagrangian relaxation)
 - c. _____ (the value of the dual objective function, i.e., the lower bound.)
 - d. _____ (the value of the third Lagrangian multiplier)
 - e. _____ f. _____ g. _____
(the subgradient of the dual objective fn.)

d. Does this Lagrangian relaxation possess the "integrality property"? Why or why not?

e. What does your answer in (d) imply about the strength of the lower bound which can be obtained from this relaxation?

(5.) **The Median Plant Location Problem:** Consider the network below:



		Distances							
		to							
		1	2	3	4	5	6	7	8
f r o m	1	0	22	29	999	999	999	999	30
	2	22	0	12	21	999	59	73	36
	3	29	12	0	12	13	999	999	999
	4	999	21	12	0	9	49	999	999
	5	999	999	13	9	0	999	999	999
	6	999	59	999	49	999	0	69	999
	7	999	73	999	999	999	69	0	999
	8	30	36	999	999	999	999	999	0

i	1	2	3	4	5	6	7	8
X[i]	35	54	64	75	76	95	32	21
Y[i]	98	86	93	88	97	43	16	71
Wt[i]	7	6	3	4	8	4	4	4

Floyd's algorithm was applied to find the following matrix of shortest path lengths (and the predecessors of nodes on the shortest paths):

		Shortest Path Lengths							
		to							
		1	2	3	4	5	6	7	8
f r o m	1	0	22	29	41	42	81	95	30
	2	22	0	12	21	25	59	73	36
	3	29	12	0	12	13	61	85	48
	4	41	21	12	0	9	49	94	57
	5	42	25	13	9	0	58	98	61
	6	81	59	61	49	58	0	69	95
	7	95	73	85	94	98	69	0	109
	8	30	36	48	57	61	95	109	0

		Predecessor Lists							
		to							
		1	2	3	4	5	6	7	8
f r o m	1	0	1	1	3	3	2	2	1
	2	2	0	2	2	3	2	2	2
	3	3	3	0	3	3	4	2	2
	4	3	4	4	0	4	4	2	2
	5	3	3	5	5	0	4	2	2
	6	2	6	4	6	4	0	6	2
	7	2	7	2	2	3	7	0	2
	8	8	8	2	2	3	2	2	0

Then the matrix of weighted shortest path lengths was computed:

		Weighted Shortest Path Lengths							
		to							
		1	2	3	4	5	6	7	8
f r o m	1	0	132	87	164	336	324	380	120
	2	154	0	36	84	200	236	292	144
	3	203	72	0	48	104	244	340	192
	4	287	126	36	0	72	196	376	228
	5	294	150	39	36	0	232	392	244
	6	567	354	183	196	464	0	276	380
	7	665	438	255	376	784	276	0	436
	8	210	216	144	228	488	380	436	0

- a. Formulate the 3-median problem as a binary integer LP.

The addition/substitution heuristic was applied to try to find the 3-median set, giving the output below:

K-median
Facility Location
Problem

1-Median:

2

Cost = **a**

2-Median:

Addition: 2 **b**

Cost: 854

No substitution can be made

3-Median:

Addition: 2 7 4

Cost: **c**

Substitution: **d** 7 4

Cost: 550

- b. Four values are blanked in the output of the addition/substitution heuristic. What are these values?
- a. _____ (the cost of the 1-median set {2})
 - b. _____ (the facility added to the 1-median set {2})
 - c. _____ (the cost of the set of facilities: 2, 7, &4)
 - d. _____ (the facility substituted for #2)

c. Find the vertex **center** for this network (with unweighted distances).

d. Define the objective function which is being minimized in the problem of finding the absolute center of the network.

e. Below is some output displaying a lower bound which may be computed for the center objective function on each edge. What is the missing value?

Lower Bounds on Local Edge-Centers

i	j	LB
2	6	54.5
2	7	54.5
6	7	67.5
4	6	70
1	2	73
2	3	73
2	4	73
2	8	73
1	3	[]
3	4	83.5
3	5	85
1	8	87
4	5	91.5

f. Which edges can be eliminated from consideration when searching for the absolute center?

g. Below is information about the center objective function on the edge (1,3). What are the three missing values?

The Objective Function on edge [1,3]

Monotonically increasing distance functions: $d(x,k)$ where

k=	1
$d(i,k)=$	0
$d(j,k)=$	29

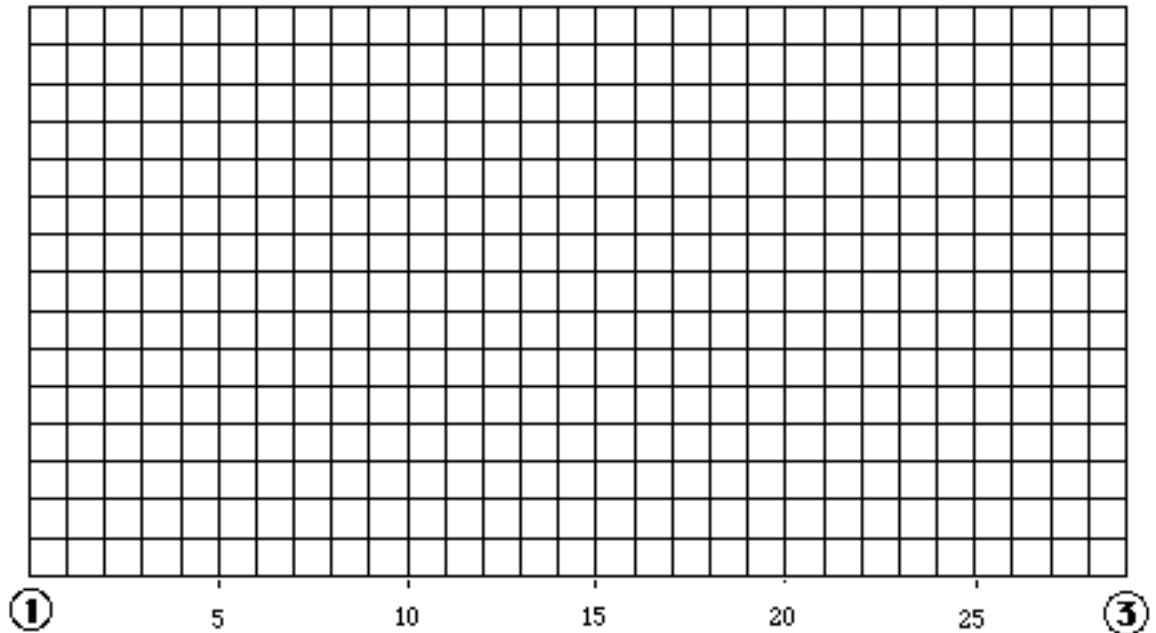
Monotonically decreasing distance functions: $d(x,k)$ where

k=	3	4	5
$d(i,k)=$	29	41	42
$d(j,k)=$	0	12	[]

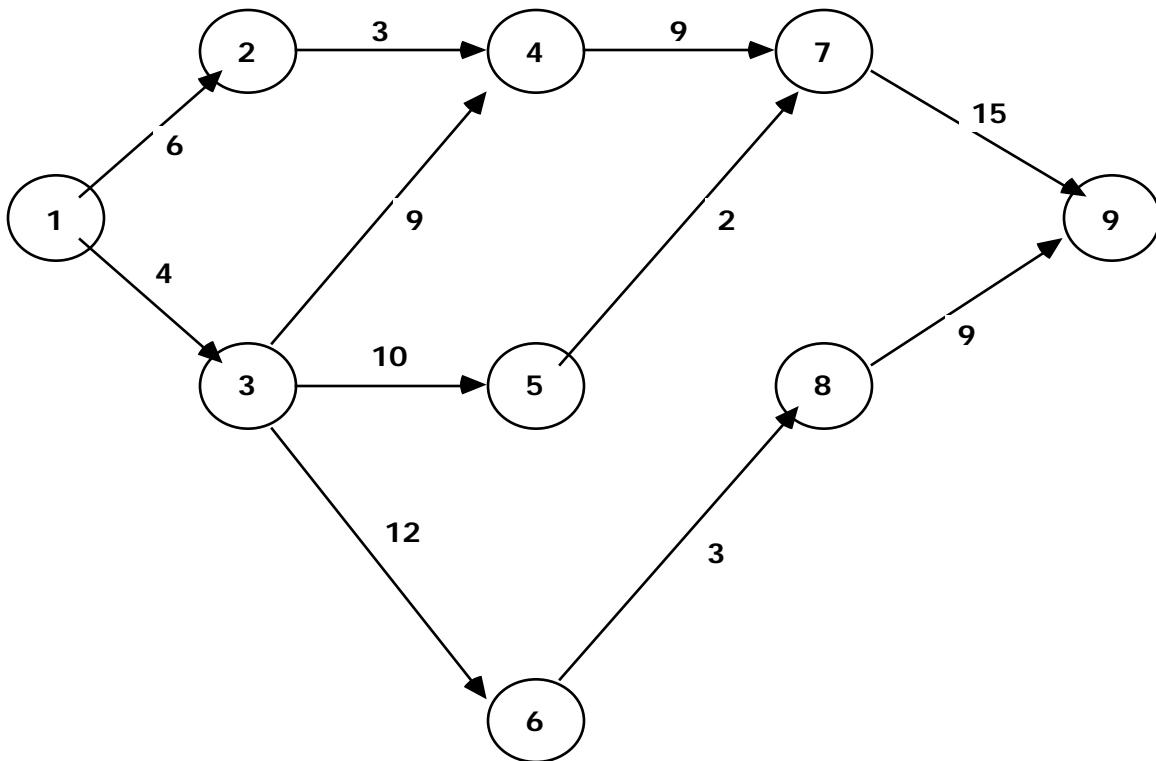
Distance functions which increase to a peak at a point Δ units from i, then decrease: $d(x,k)$ where

k=	2	6	7	8
$d(i,k)=$	22	81	95	30
$d(j,k)=$	12	61	85	[]
$\Delta=$	[]	4.5	9.5	23.5

h. Sketch the center objective function on the edge (1,3). What is the edge center of the edge (1,3)?



6. **Project Scheduling:** Consider the project consisting of eleven activities, represented by the AOA (activity-on-arrow) diagram below:



- a. For each activity, compute Earliest Start Time.
- b. For each activity, compute Earliest Finish Time

Write the values for (a) and (b) directly on the diagram.

- c. What activities are on the critical path? (*Indicate the path on the diagram.*)
- d. What is the earliest that the project can be completed, if it is begun at time zero? _____
- e. What is the Total Float (or slack) of activity (3,4)? _____ of activity (3,5)? _____
- f. What is the Total Float of an activity on the critical path? _____

7. Integer Programming Models Dorian Auto is considering manufacturing three types of autos: compact, midsize, and large. The resources required for, and the profits yielded by, each type of car are as follows:

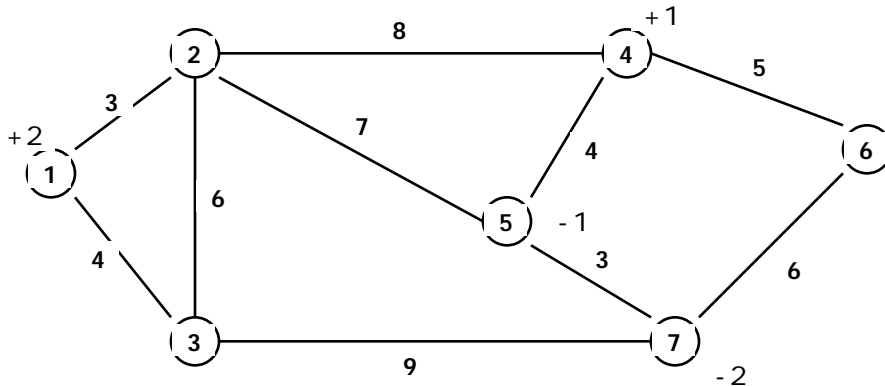
	<u>COMPACT</u>	<u>MIDSIZE</u>	<u>LARGE</u>
Steel Req'd.	1.5 tons	3.0 tons	5.0 tons
Labor Req'd.	30.0 hrs	25.0 hrs	40.0 hrs
Profit yielded	\$2000	\$3000	\$4000
Setup Cost	\$50000	\$80000	\$100000

The Setup cost (for design, tooling, etc.) is incurred if that type of car is to be produced. At present 6000 tons of steel and 60,000 hrs of labor are available. In order for production of a type of car to be economically feasible, management has specified that at least 1000 cars of that type must be produced. Use the following variables:

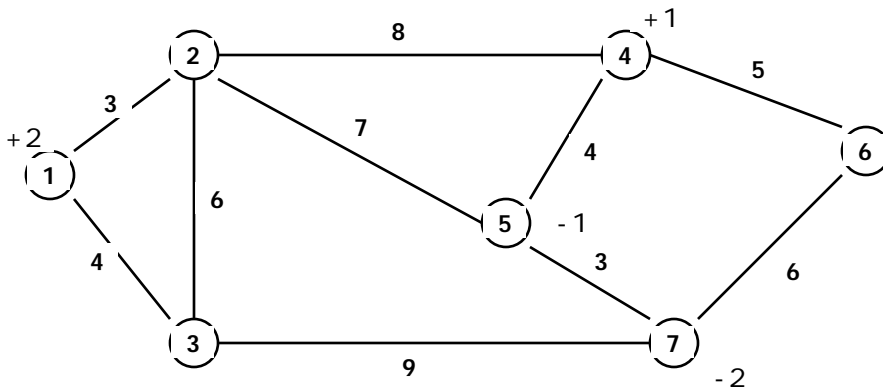
- X_C = # of compact cars to be produced
 X_M = # of midsize cars to be produced
 X_L = # of large cars to be produced
 Y_C = 1 if compact cars are to be produced, else 0
 Y_M = 1 if midsize cars are to be produced, else 0
 Y_L = 1 if large cars are to be produced, else 0

- a. Formulate an integer linear programming model to maximize Dorian's profit.
- b. Add a constraint which would specify that if midsize cars are produced, then compacts must also be produced.
- c. Add a constraint which would specify that either compacts or large cars (or both) must be produced.

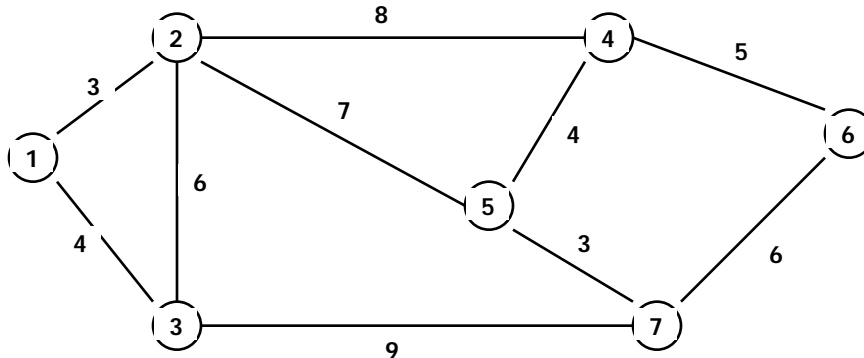
8. Primal Simplex Algorithm for Networks. Consider the network below, where the number alongside each node represents supply or demand, i.e., node #1 has a supply of 2 units of a commodity, node #4 has 1 unit, node #5 requires 1 unit, and node #7 requires 2 units. The numbers alongside the arcs represent unit shipping costs.



- Find the minimum spanning tree of this network, and indicate it above.
- Using the minimum spanning tree (plus artificial "root" arc) as an initial basis, compute the corresponding basic solution, i.e., flows. Indicate these flows below:

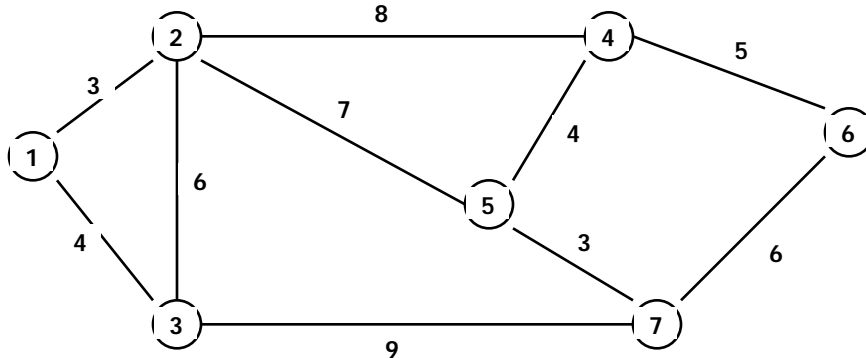


- Using the same basis, compute the dual variables (simplex multipliers), and indicate below:

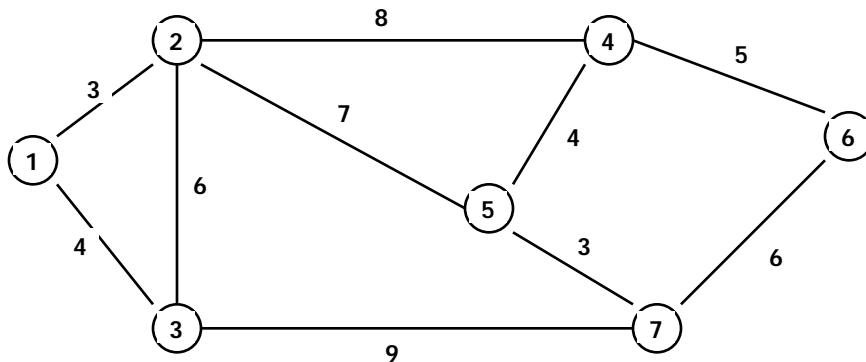


- Choose one arc not in the rooted spanning tree, and "price" it, i.e., compute its reduced cost. Should this arc enter the basis or not?

- e. Regardless of whether the arc you selected in (d) should enter the basis, explain how to enter the arc into the basis and how to choose the arc leaving the basis. Indicate the new basis on the network below:



9. **Postman Problem.** Consider the street network given below, where the numbers alongside the streets are the lengths, in hundreds of feet. Suppose that a postman must deliver mail to houses along all these streets. (Assume that he can deliver to both sides of the streets simultaneously.)



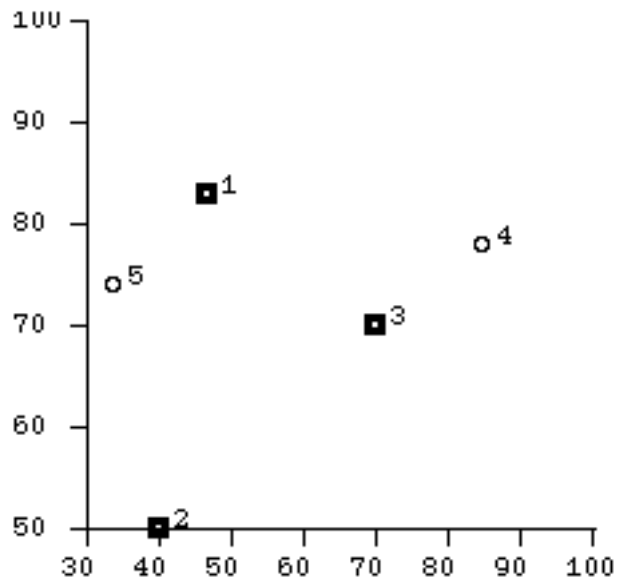
- Why can't this be done without traversing some street(s) more than once?
- What streets should be traversed more than once?

10. Lagrangian Relaxation: Consider the following plant location problem:

Number of sources = $M = 3$
 Number of destinations = $N = 5$
 Total demand: 21

Costs, Supplies, Demands							
i/j	1	2	3	4	5	K	F
1	0	30	30	40	20	11	5000
2	30	0	40	50	30	8	2000
3	30	40	0	20	40	9	3000
Demand:	6	2	9	2	2	28	0

K = capacity,
 F = fixed cost



a. Formulate the problem as a mixed-integer LP problem:

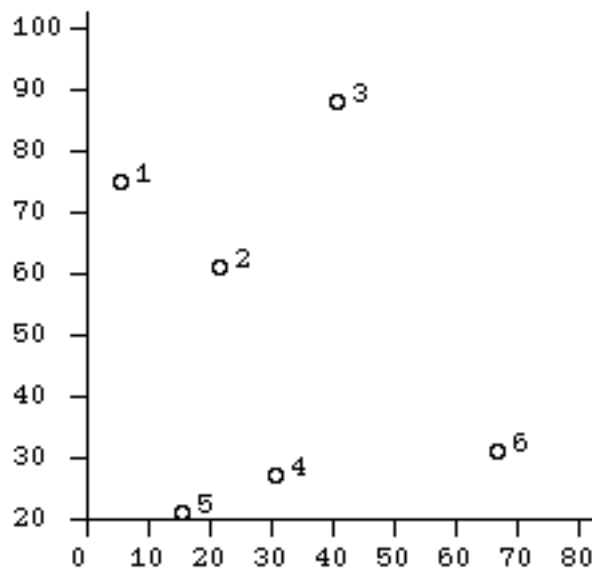
b. Apply Lagrangian relaxation to the supply constraints of your formulation. Write the Lagrangian subproblem:

c. Letting the Lagrange multipliers be **each** equal to 10, illustrate how the Lagrangian subproblem is easily solved.

d. What is the objective value of the Lagrangian subproblem? Is it a lower or upper bound on the optimum of the original problem?

e. For each of the Lagrange multipliers, specify whether it should be adjusted upward or downward in order to improve the bound.

11. Vehicle Routing Problem: Consider the problem below, where a truck depot is located in city #6. Each truck (with capacity 9 units of weight), and deliveries are to be made to the other five cities, as shown below. (Ignore the delivery to be made in city #6.)



i	1	2	3	4	5	6
X[i]	6	22	41	31	16	67
Y[i]	75	61	88	27	21	31
Wt[i]	2	9	7	3	4	2

Vehicle capacity = 9, with depot at node 6
 Total weight of nodes (excluding depot): 25
 Average weight per node: 5

Distances

```

-----
|
f|
r|
o| 1  2  3  4  5  6
m|---
1| 0 21 37 54 55 75
2|21 0 33 35 40 54
3|37 33 0 62 72 63
4|54 35 62 0 16 36
5|55 40 72 16 0 52
6|75 54 63 36 52 0
    
```

a. Choose a non-zero entry of the savings matrix below and explain how it is computed:

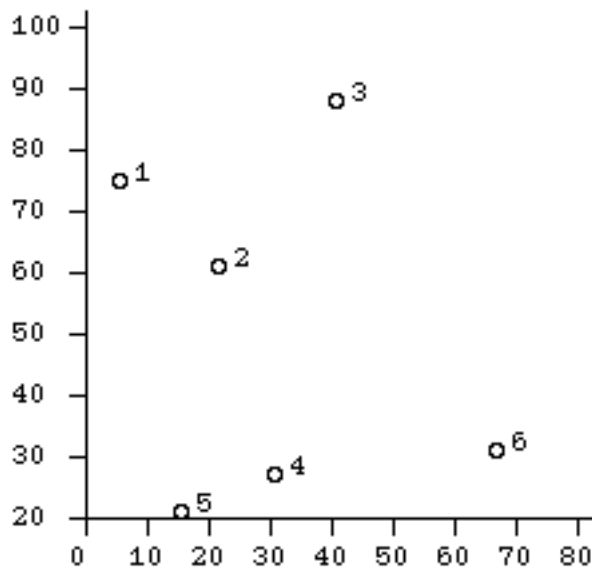
Savings Matrix

```

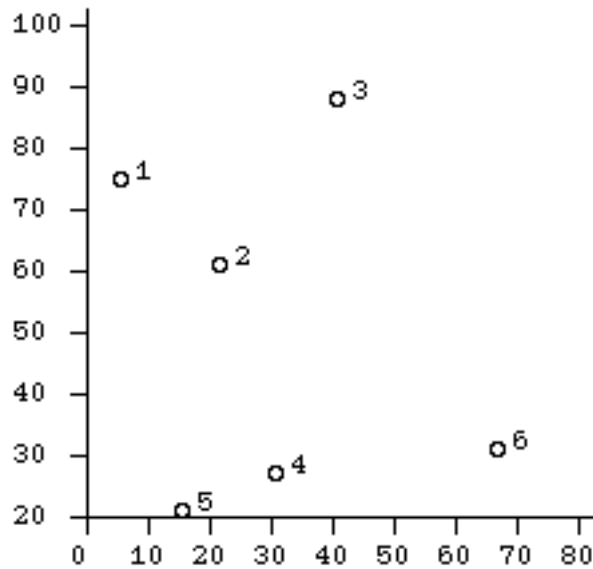
-----
f|
r|
o| 1  2  3  4  5  6
m|---
1|  0 108 101 57 72 0
2|108  0 84 55 66 0
3|101 84  0 37 43 0
4| 57 55 37  0 72 0
5| 72 66 43 72  0 0
6|  0  0  0  0  0 0
    
```

Next you will perform the computations of the Clark-Wright heuristic to find a set of delivery routes.

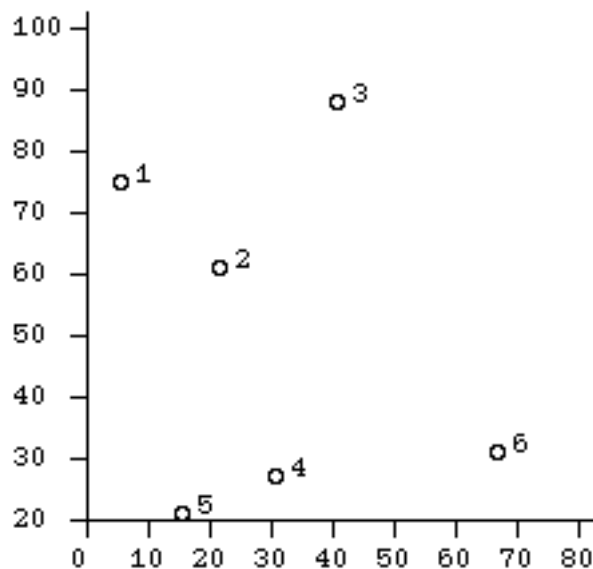
b. Indicate the result of the **first** step below:



c. Indicate the result of the **second** step below:



d. Do a **third** step, and indicate the results below:



12. Formulation of Plant Location Problem: Four possible locations of plants are being considered, to supply demand of customers in nine cities. One or more plant locations are to be selected. Let

D_j = annual demand of city j ($j=1,2,\dots,9$)

K_i = annual production capacity of plant at location i , *if built* ($i=1,2,3,4$)

c_{ij} = shipping cost (per unit shipped) between plant i and city j

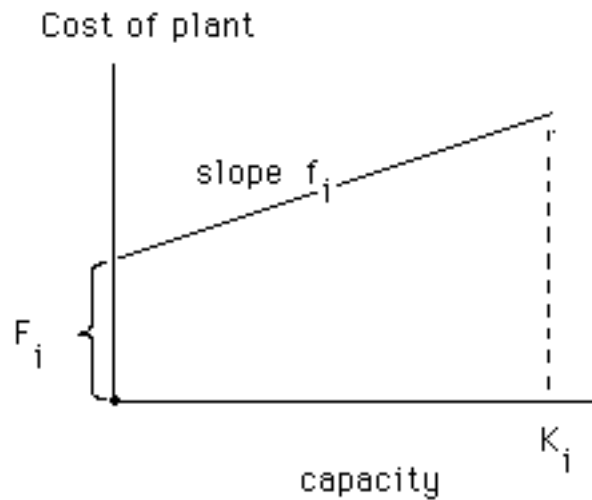
F_i = annual cost of capital to build & operate a plant at location i

- a. Formulate a mixed-integer linear model to choose the locations where plants should be built, using a binary variable Y_i to indicate selection of plant location i . How many integer variables & continuous variables are required? How many constraints are required?

- b. Write constraints for each of the additional restrictions:
- Plant 3 should not be built unless either plant 1 or plant 2 is also built.

ii.) No more than 3 plants should be built.

- c. Reformulate the model in part (a) so that the capacity of a plant at location i may be any value *between* zero and K_i (inclusive), and that the annual cost to build & operate the plant depends upon the capacity which is selected, as indicated in the graph:



- d. Reformulate the model in part (c) so that each plant i should not be built unless it is to produce & ship at least a minimum quantity L_i ($0 < L_i < K_i$)