56:272 Integer Pgmg & Network Flows
Final Exam -- December 12, 1990
Instructor: D. L. Bricker

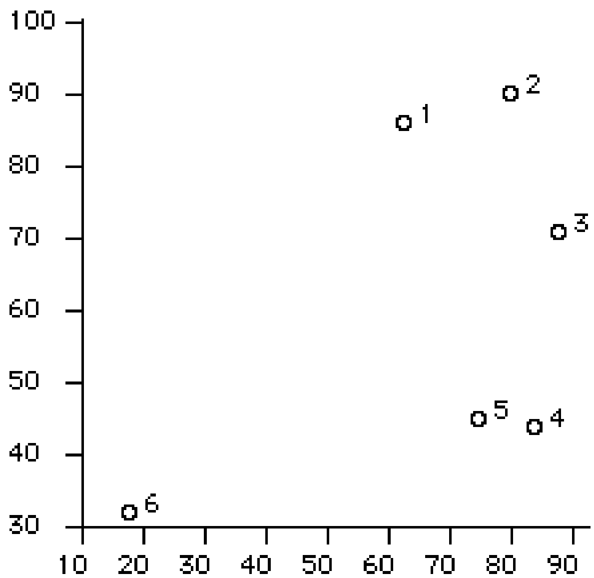
Answer 1 (out of 2) of PART ONE, plus 3 (out of 4) questions from PART TWO.

PART ONE
- ______ 1. Traveling salesman problem
- ______ 2. Assignment problem

PART TWO
- ______ 3. Gomory's cutting plane algorithm
- ______ 4. Benders' partitioning algorithm
- ______ 5. Generalized assignment problem
- ______ 6. Location problem in network

❑ ❑ ❑ PART ONE ❑ ❑ ❑

(1.) Traveling Salesman Problem Consider the 5 cities below, which are each to be visited by a vehicle which will begin and end its route at city #1. *(Warning: the horizontal & vertical axes are scaled differently!)*



i	1	2	3	4	5	6
X[i]	63	80	88	84	75	18
Y[i]	86	90	71	44	45	32

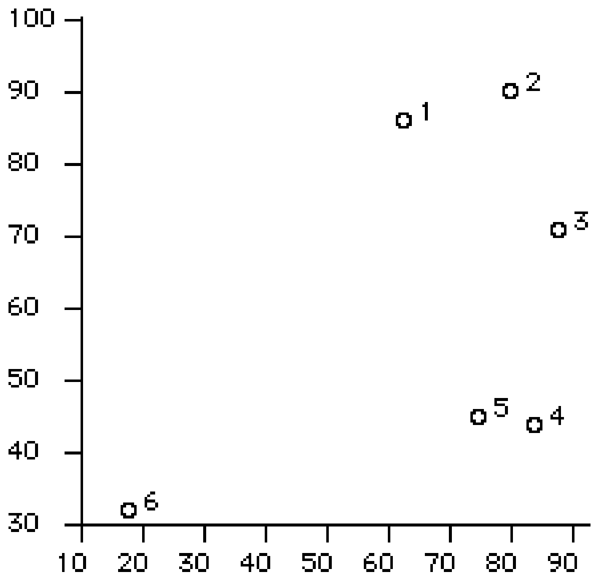
Distances

from \ to	1	2	3	4	5	6
1	0	17	29	47	43	70
2	17	0	21	46	45	85
3	29	21	0	27	29	80
4	47	46	27	0	9	67
5	43	45	29	9	0	58
6	70	85	80	67	58	0

a. Apply the nearest neighbor heuristic (starting at node 1) to the problem. What is the length of the tour?

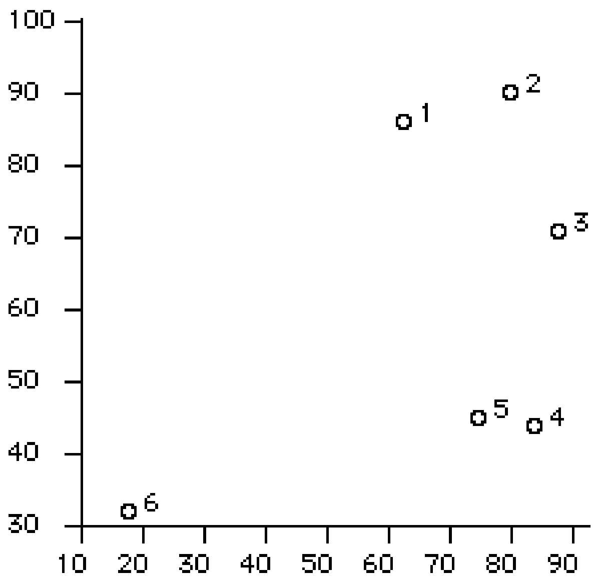
b. Find the minimum spanning one-tree of the nodes above (letting node #1 be the "root" node). Is it a tour?

c. Assign vertex penalties and perform one iteration of the vertex penalty method, using a "unit penalty" of 10. Does it result in a tour? (Indicate the result below.)



d. Explain how the vertex penalty method may be interpreted as a Lagrangian relaxation method for the traveling salesman problem. What constraints are being "relaxed"? What is the objective function of the Lagrangian relaxation?

e. Perform a second iteration of the vertex penalty method. Is the result a tour?



f. Based upon your answers above, state an **upper** and a **lower** bound on the length of the optimal tour.

g. Suppose that the Hungarian algorithm were applied to the distance matrix of a TSP. Does the solution always satisfy the constraint relaxed in (d)?

(2.) Assignment Problem: Consider the problem of assigning 5 jobs to 5 machines, with the following cost matrix (where the element in row i & column j is the cost of assigning job i to machine j) :

1	5	3	4	10
3	8	5	1	8
10	9	5	9	3
9	5	8	2	10
2	10	1	6	7

a. Formulate this problem as a binary integer LP.

b. If you apply the simplex algorithm to the LP relaxation of this problem, are you guaranteed of obtaining an integer solution? Why?

c. How many basic variables does the LP relaxation of this problem have? How many are positive at the optimum? Such a basic solution of an LP has the property called _________________.

d. Apply the row reduction step of the Hungarian algorithm to this matrix, and then the column reduction step, completing the matrices below

row reduction:

0	4	2	3	9
2	7	4	0	7
7		2	6	0
7	3	6	0	8
1	9		5	6

column reduction:

0	1	2	3	9
2		4	0	7
7	3	2	6	0
7	0	6	0	8
1	6	0		6

e. Check the resulting matrix for optimality. Are any further steps required? If so, perform them. If not, what is the optimal solution?

f. Write the dual constraints of the LP relaxation of the assignment problem. Show that the values by which the rows & columns were reduced in (d) satisfy these constraints, and the complementary slackness conditions.

g. How do the generalized assignment problem and the quadratic assignment problem differ from the assignment problem such as was solved above? (How do the models differ mathematically? Describe differences in the applications which the three formulations might model.)

❑ ❑ ❑ PART TWO ❑ ❑ ❑

(3.) GOMORY'S FRACTIONAL CUTTING-PLANE ALGORITHM

Consider the problem:

Maximize $3X_1 - X_2$

subject to $3X_1 - 2X_2 \leq 3$

$5X_1 + 4X_2 \geq 10$

$2X_1 + X_2 \leq 5$

$X_1, X_2 \geq 0$ & integer

After adding slack & surplus variables X_3, X_4 and X_5, and solving the LP relaxation, we get the optimal tableau:

Current LP Tableau

z	1	2	3	4	5	B
1	0	0	¯0.714	0	¯0.429	¯4.29
0	1	0	0.143	0	0.286	1.86
0	0	1	¯0.286	0	0.429	1.29
0	0	0	¯0.429	1	3.14	4.43

Variables:
(Negative of) objective function value: z
Original structural variables: 1 2
Original slack/surplus variables: 3 4 5

a. A constraint was added to the problem to exclude this optimal LP solution without excluding any integer feasible solution, using row #4 as the "source row":

```
Source row is # 4
```

i:	3	4	5	6	rhs
Source row:	¯0.429	1	3.14	0	4.43
Cut:	**a**	0	¯0.143	1	¯0.429

```
(X[6] (= slack variable for new cut)
      is basic but < 0)
```

What is the value which is blocked out above? **a** = _____________________

b. Te constraint in (a) may be expressed in terms of the original variables X_1 & X_2:

```
The cut which is added is
(in terms of original variables):
```

1	2		b
2	¯1	≤	**b**

i.e., $2X_1 - X_2 \leq$ **b** = ______________________

We next append the new constraint to the tableau above:

z	1	2	3	4	5	6	B
1	0	0	¯0.714	0	¯0.429	0	¯4.29
0	1	0	0.143	0	0.286	0	1.86
0	0	1	¯0.286	0	0.429	0	1.29
0	0	0	¯0.429	1	3.14	0	4.43
0	0	0	[illegible]	0	¯0.143	1	¯0.429

```
Variables:
     (Negative of) objective function value: z
     Original structural variables:    1 2
     Original slack/surplus variables: 3 4 5
     Slack variables for cuts:         6
```

and perform a dual simplex pivot to regain feasibility:

z	1	2	3	4	5	6	B
1	0	0	0	0	¯0.25	¯1.25	¯3.75
0	1	0	0	0	0.25	0.25	1.75
0	0	1	0	0	0.5	¯0.5	**c**
0	0	0	0	1	3.25	¯0.75	4.75
0	0	0	1	0	0.25	¯1.75	**d**

(c.) What are the values blanked out in the tableau above?

c = ___________, **d**= __________

(d.) Why is the dual, rather than the primal (i.e., the ordinary) simplex method used to re-optimize after adding the new constraint?

(4.) Benders' Decomposition Algorithm for Plant Location: Consider the following randomly-generated problem in which demand in 8 cities is to be satisfied by building plants in one or more of four of the cities:

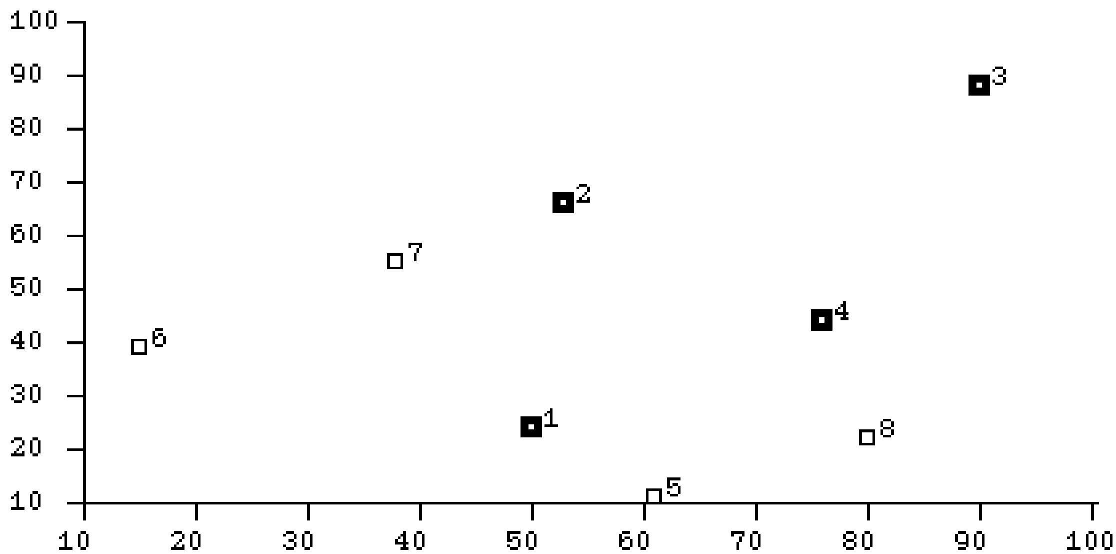
Random Problem (Seed = 3723316)

Number of sources = M = 4
Number of destinations = N = 8
Total demand: 40

Costs, Supplies, Demands

i/j	1	2	3	4	5	6	7	8	K	F
1	0	42	75	33	17	38	33	30	19	498
2	42	0	43	32	56	47	19	52	15	23
3	75	43	0	46	82	90	62	67	17	89
4	33	32	46	0	36	61	40	22	17	129
Demand:	6	1	2	5	5	8	3	10	68	0

K = capacity,
F = fixed cost
<READY?>



Plant Location Problem

(a.) State the mixed-integer programming formulation of the problem. How many continuous variables (X) and how many binary (zero-one) variables (Y) are required?

(b.) Give the expression for the optimal value as a function of Y, i.e. v(Y), expressed in terms of X.

A trial solution was evaluated, in which all four plants are to be open. The result was:

Solution of Transportation Problem

```
Please enter plant sites to be open
□:
      1 2 3 4
Total capacity exceeds total demand.  Dummy demand point added.
... solving transportation problem ...

Minimum transport cost = 666
Fixed cost of plants = 739
       Total = 1405
CPU time = 9.6 sec.
Generated support αY+b, where α = 498 23 89 [      ] , b = [      ]
This is support # 1

*** New incumbent! *** (replaces 10000000000)
```

Optimal Shipments

from \ to	1	2	3	4	5	6	7	8	9
1	6	0	0	0	5	8	0	0	0
2	0	1	0	0	0	0	3	0	11
3	0	0	2	0	0	0	0	0	15
4	0	0	0	5	0	0	0	10	2

(Demand pt #9 is dummy demand for excess capacity.)

NOTE: Solution is degenerate!

Dual Solution of Transportation Problem

```
Supply constraints: U = 0 0 0 0 0
Demand constraints: V = 0 0 0 0 17 38 19 22 0
```

Reduced costs: COST - U°.+V

0	42	75	33	0	0	14	8
42	0	43	32	39	9	0	30
75	43	0	46	65	52	43	45
33	32	46	0	19	23	21	0

(c.) What are the values of the two coefficients of the linear support which are blanked out above?

(d.) Why is the dual solution not unique? How might additional linear supports be computed, using this fact?

(e.) Why is the solution to the transportation problem labeled as "degenerate"?

Next the Master Problem is solved:

```
                                    Master Problem
                                    --------------

Trial set of plants: 1 2 3
with estimated cost 1276 < incumbent ( = 1405)

Current status vectors for Balas' additive algorithm:
    j:      1  3  2 ¯4
underline:  0  0  0  0
```

Using this solution of the master problem (which was sub-optimized), the subproblem, i.e. transportation problem, was next solved:

```
                                   Solution of
                               Transportation Problem

Please enter plant sites to be open
□:
      1 2 3
Total capacity exceeds total demand.  Dummy demand point added.
... solving transportation problem ...

Minimum transport cost = 1078
Fixed cost of plants = [      ]
       Total = [      ]
CPU time = 20.9 sec.
Generated support αY+b, where α = 688 503 871 129, b = ¯374
This is support # 2
```

(f.) What are the two values blanked out above?

Next the master problem is sub-optimized again:

```
                                    Master Problem

Trial set of plants: 1 3 4

with estimated cost [      ] < incumbent ( = 1405)

Current status vectors for Balas' additive algorithm:
    j:      1  3 ¯2  4
underline:  0  0  1  0
```

(f.) What is the value blanked out above?

(g.) Sketch the enumeration tree, showing the node corresponding to the solution of the master problem above, and indicate parts of the tree which have been fathomed.

Next the subproblem was solved, using the set of plants {1, 3, 4}:

```
Solution of
Transportation Problem

Please enter plant sites to be open
⎕:
      1 3 4
Total capacity exceeds total demand.  Dummy demand point added.
... solving transportation problem ...

Minimum transport cost = 794
Fixed cost of plants = 716
       Total = 1510
CPU time = 13.05 sec.
Generated support αY+b, where α = 536 23 820 486, b = ¯332
This is support # 3
```

(h.) Can Benders' algorithm be terminated at this iteration? Explain why or why not.

(i.) Suppose that we wish to estimate the cost of the proposal to open plants #2,3,&4. How can this be done using $\underline{v}_3$? Does this give us an over- or under-estimate of the cost? Could this set of plants possibly be optimal?

(j.) Suppose at some node of the enumeration tree, the "status vector" J is {$\underline{2}$, -1, $\underline{4}$}. What would be the next node to be considered if this node is fathomed? (Give the value of J.)

(5.) Generalized Assignment Problem: Consider the problem of assigning 4 jobs to 3 machines (each with limited capacity):

Costs

Machine i	Job 1	Job 2	Job 3	Job 4
1	11	14	13	21
2	12	23	17	20
3	21	14	12	14

Resources Used

Machine i	Job 1	Job 2	Job 3	Job 4	Available b
1	5	11	25	11	23
2	14	5	13	16	27
3	13	18	10	14	17

a. Formulate this problem as a binary integer programming problem.

b. Suppose that the capacity constraints are relaxed, using the Lagrangian relaxation method. The first 3 iterations of the subgradient optimization method to maximize the lower bound appears below, where the value of a feasible solution was found to be 56, and a stepsize parameter was assigned the value 0.75.

```
Lambda = 0.75
Upper bound Z* = 56
```

Iteration # 1

Multiplier vector U = 0 0 0
Objective function of relaxation:

from \ to	1	2	3	4
1	11	14	13	21
2	12	23	17	20
3	21	14	12	14

Dual value is 51
Variables selected from GUB sets are:
1 1 3 3
Resources used are: 16 0 24,
(Available: 23 27 17)
Subgradient of Dual Objective is ¯7 ¯27 7
Stepsize is 0.0765306

Iteration # 2

Multiplier vector U = 0 0 0.535714
Objective function of relaxation:

from \ to	1	2	3	4
1	11	14	13	21
2	12	**a**	17	20
3	27.9643	23.6429	17.3571	**b**

Dual value is **c**
Variables selected from GUB sets are:
1 1 1 2
Resources used are: 41 16 0,
(Available: 23 27 17)
Subgradient of Dual Objective is 18 ¯11 ¯17
Stepsize is 0.00869553

Iteration # 3

Multiplier vector U = 0.156519 0 **d**
Objective function of relaxation:

from \ to	1	2	3	4
1	11.7826	15.7217	16.913	22.7217
2	12	23	17	20
3	26.0426	20.982	15.8789	19.4305

Dual value is 52.6196
Variables selected from GUB sets are:
1 1 3 3
Resources used are: 16 0 24,
(Available: 23 27 17)
Subgradient of Dual Objective is
e **f** **g**
Stepsize is 0.0223599

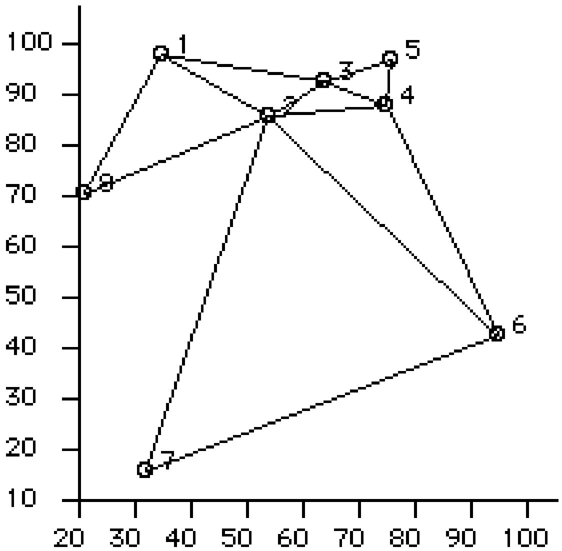
c. Several values have been omitted from the output. Compute their values:

a. _______________ (cost coefficient of the Lagrangian relaxation)
b. _______________ (cost coefficient of the Lagrangian relaxation)
c. _______________ (the value of the dual objective function, i.e., the lower bound.)
d. _______________ (the value of the third Lagrangian multiplier)
e. _______________ **f**. _______________ **g**. _______________
(the subgradient of the dual objective fn.)

d. Does this Lagrangian relaxation possess the "integrality property"? Why or why not?

e. What does your answer in (d) imply about the strength of the lower bound which can be obtained from this relaxation?

(6.) The Median Plant Location Problem: Consider the network below:



Distances

from \ to	1	2	3	4	5	6	7	8
1	0	22	29	999	999	999	999	30
2	22	0	12	21	999	59	73	36
3	29	12	0	12	13	999	999	999
4	999	21	12	0	9	49	999	999
5	999	999	13	9	0	999	999	999
6	999	59	999	49	999	0	69	999
7	999	73	999	999	999	69	0	999
8	30	36	999	999	999	999	999	0

i	1	2	3	4	5	6	7	8
X[i]	35	54	64	75	76	95	32	21
Y[i]	98	86	93	88	97	43	16	71
Wt[i]	7	6	3	4	8	4	4	4

Floyd's algorithm was applied to find the following matrix of shortest path lengths (and the predecessors of nodes on the shortest paths):

Shortest Path Lengths

from \ to	1	2	3	4	5	6	7	8
1	0	22	29	41	42	81	95	30
2	22	0	12	21	25	59	73	36
3	29	12	0	12	13	61	85	48
4	41	21	12	0	9	49	94	57
5	42	25	13	9	0	58	98	61
6	81	59	61	49	58	0	69	95
7	95	73	85	94	98	69	0	109
8	30	36	48	57	61	95	109	0

Predecessor Lists

from \ to	1	2	3	4	5	6	7	8
1	0	1	1	3	3	2	2	1
2	2	0	2	2	3	2	2	2
3	3	3	0	3	3	4	2	2
4	3	4	4	0	4	4	2	2
5	3	3	5	5	0	4	2	2
6	2	6	4	6	4	0	6	2
7	2	7	2	2	3	7	0	2
8	8	8	2	2	3	2	2	0

Then the matrix of weighted shortest path lengths was computed:

Weighted Shortest Path Lengths

from \ to	1	2	3	4	5	6	7	8
1	0	132	87	164	336	324	380	120
2	154	0	36	84	200	236	292	144
3	203	72	0	48	104	244	340	192
4	287	126	36	0	72	196	376	228
5	294	150	39	36	0	232	392	244
6	567	354	183	196	464	0	276	380
7	665	438	255	376	784	276	0	436
8	210	216	144	228	488	380	436	0

a. Formulate the 3-median problem as a binary integer LP.

The addition/substitution heuristic was applied to try to find the 3-median set, giving the output below:

```
K-median
Facility Location
Problem

1-Median:

2
   Cost = [ a ]

2-Median:

Addition: 2 [ b ]
   Cost: 854
No substitution can be made

3-Median:

Addition: 2 7 4
   Cost: [ c ]

Substitution: [d] 7 4
   Cost: 550
```

b. Four values are blanked in the output of the addition/substitution heuristic. What are these values?
a. ____________ (the cost of the 1-median set {2})
b. ____________ (the facility added to the 1-median set {2})
c. ____________ (the cost of the set of facilities: 2, 7, &4)
d. ____________ (the facility substituted for #2)

c. Find the vertex **center** for this network (with unweighted distances).

d. Define the objective function which is being minimized in the problem of finding the absolute center of the network.

e. Below is some output displaying a lower bound which may be computed for the center objective function on each edge. What is the missing value?

Lower Bounds on Local Edge-Centers

i	j	LB
2	6	54.5
2	7	54.5
6	7	67.5
4	6	70
1	2	73
2	3	73
2	4	73
2	8	73
1	3	
3	4	83.5
3	5	85
1	8	87
4	5	91.5

f. Which edges can be eliminated from consideration when searching for the absolute center?

g. Below is information about the center objective function on the edge (1,3). What are the three missing values?

The Objective Function on edge [1,3]

Monotonically increasing distance functions: d(x,k) where

k=	1
d(i,k)=	0
d(j,k)=	29

Monotonically decreasing distance functions: d(x,k) where

k=	3	4	5
d(i,k)=	29	41	42
d(j,k)=	0	12	

Distance functions which increase to a peak at a point Δ units from i, then decrease: d(x,k) where

k=	2	6	7	8
d(i,k)=	22	81	95	30
d(j,k)=	12	61	85	
Δ=		4.5	9.5	23.5

h. Sketch the center objective function on the edge (1,3). What is the edge center of the edge (1,3)?