56:272 Integer Pgmg. & Network Flows
FINAL Examination
December 9, 1989

Select 8 of the 10 problems.

Problem:	1	2	3	4	5	6	7	8	9	10	Total
Score:											

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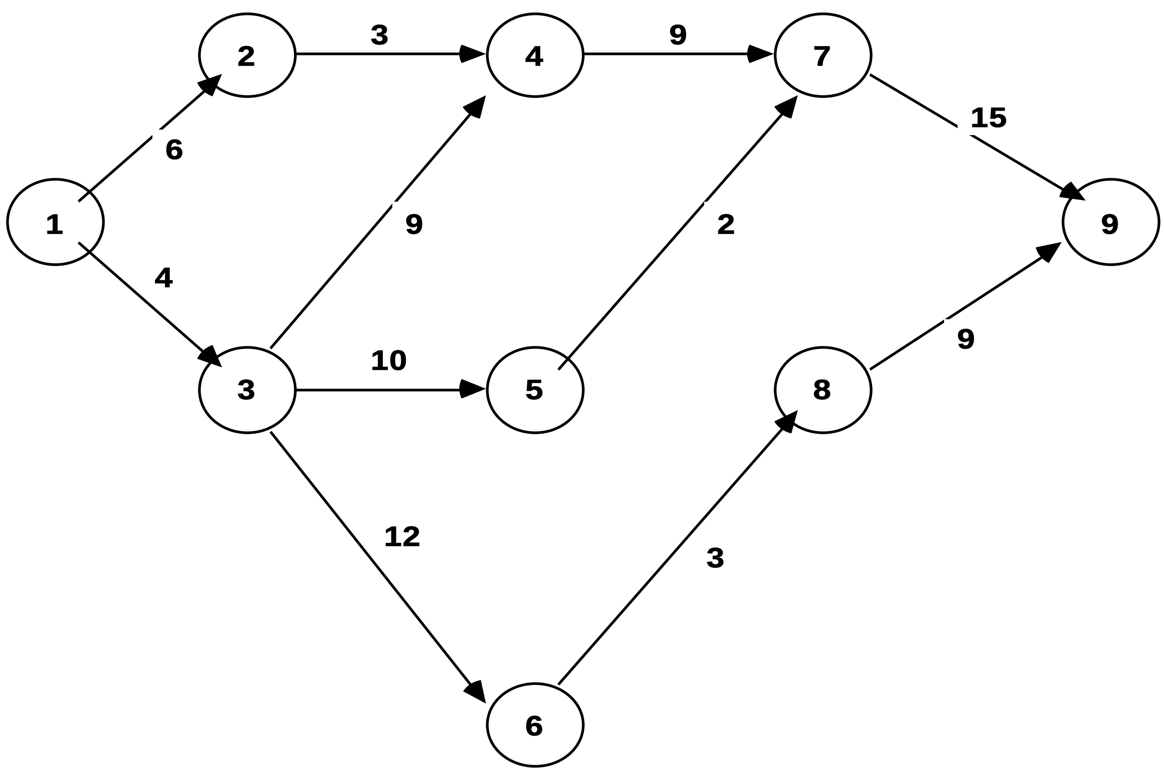
1. Project Scheduling : Consider the project consisting of eleven activities, represented by the AOA (activity-on-arrow) diagram below:



a. For each activity, compute Earliest Start Time.
b. For each activity, compute Earliest Finish Time
Write the values for (a) and (b) directly on the diagram.

c. What activities are on the critical path? (Indicate the path on the diagram.)

d. What is the earliest that the project can be completed, if it is begun at time zero? _________

e. What is the Total Float (or slack) of activity (3,4)? ________ of activity (3,5)? _________

f. What is the Total Float of an activity on the critical path? ___________

2. Integer Programming Models   Dorian Auto is considering manufacturing three types of autos: compact, midsize, and large. The resources required for, and the profits yielded by, each type of car are as follows:

	COMPACT	MIDSIZE	LARGE
Steel Reqd.	1.5 tons	3.0 tons	5.0 tons
Labor Reqd.	30.0 hrs	25.0 hrs	40.0 hrs
Profit yielded	\$2000	\$3000	\$4000
Setup Cost	\$50000	\$80000	\$100000

The Setup cost (for design, tooling, etc.) is incurred if that type of car is to be produced. At present 6000 tons of steel and 60,000 hrs of labor are available. In order for production of a type of car to be economically feasible, management has specified that <u>at least</u> 1000 cars of that type <u>must</u> be produced. Use the following variables:

X_C = # of compact cars to be produced

X_M = # of midsize cars to be produced

X_L = # of large cars to be produced

Y_C = 1 if compact cars are to be produced, else 0

Y_M = 1 if midsize cars are to be produced, else 0

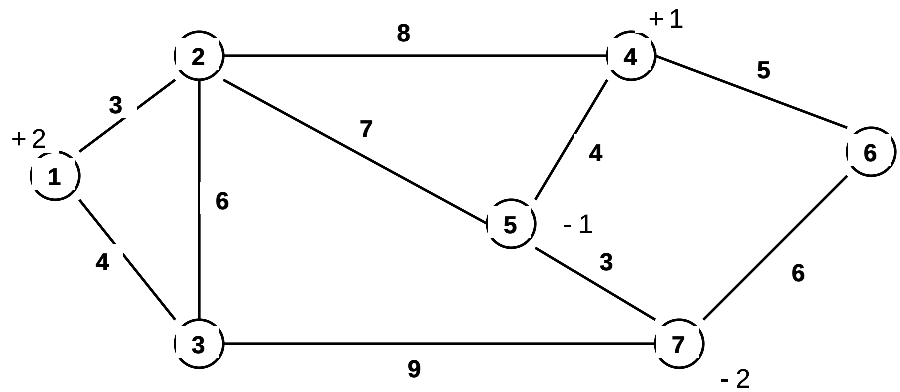
Y_L = 1 if large cars are to be produced, else 0

a. Formulate an <u>integer</u> linear programming model to maximize Dorian's profit.

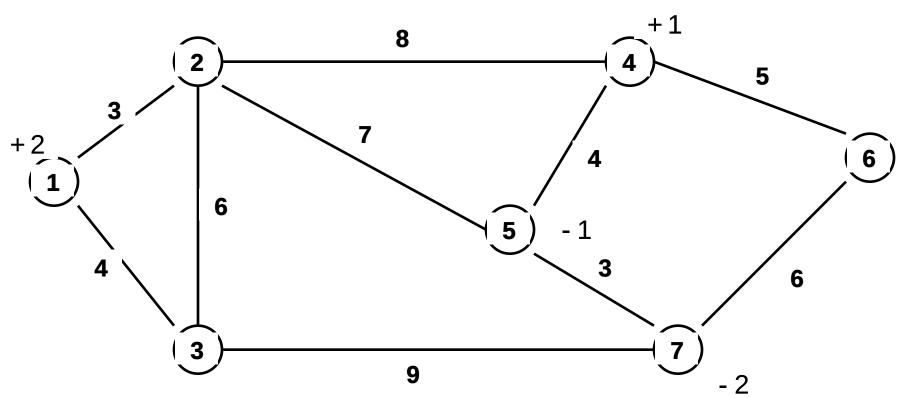
b. Add a constraint which would specify that if midsize cars are produced, then compacts must also be produced.

c. Add a constraint which would specify that either compacts or large cars (or both) must be produced.

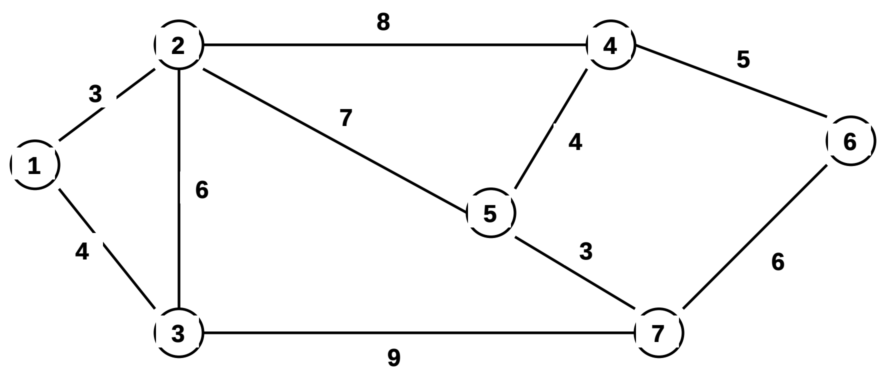
3. Primal Simplex Algorithm for Networks. Consider the network below, where the number alongside each node represents supply or demand, i.e., node #1 has a supply of 2 units of a commodity, node #4 has 1 unit, node #5 requires 1 unit, and node #7 requires 2 units. The numbers alongside the arcs represent unit shipping costs.



a. Find the minimum spanning tree of this network, and indicate it above.
b. Using the minimum spanning tree (plus artificial "root" arc) as an initial basis, compute the corresponding basic solution, i.e., flows. Indicate these flows below:

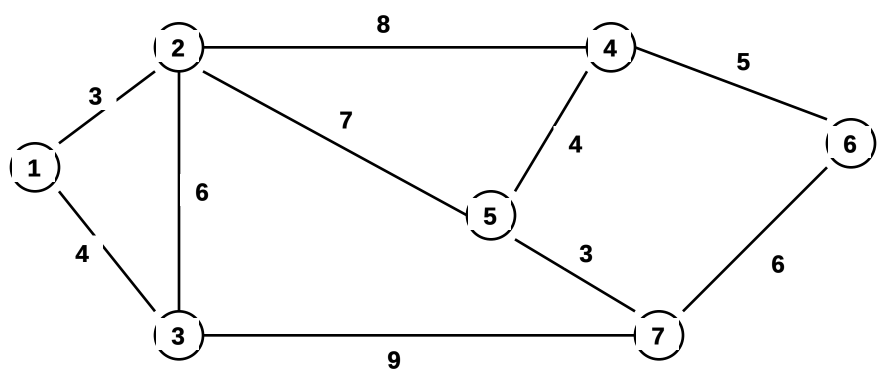


c. Using the same basis, compute the dual variables (simplex multipliers), and indicate below:

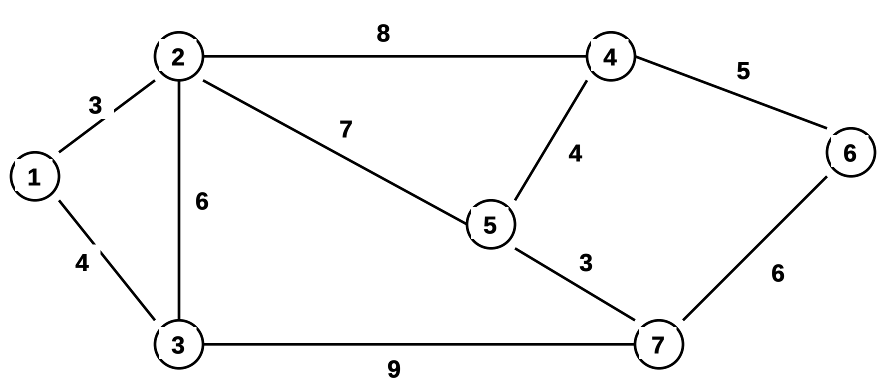


d. Choose one arc not in the rooted spanning tree, and "price" it, i.e., compute its reduced cost. Should this arc enter the basis or not?

e. Regardless of whether the arc you selected in (d) should enter the basis, explain how to enter the arc into the basis and how to choose the arc leaving the basis. Indicate the new basis on the network below:



4. Postman Problem . Consider the street network given below, where the numbers alongside the streets are the lengths, in hundreds of feet. Suppose that a postman must deliver mail to houses along all these streets. (Assume that he can deliver to both sides of the streets simultaneously.)



a. Why can't this be done without traversing some street(s) more than once?

b. What streets should be traversed more than once?

c. Although the optimization can be done by inspection in this simple case, formulate an integer programming model to choose the "deadheading" paths to be traveled by the postman.

5. Lagrangian Relaxation : Consider the following plant location problem:

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Number of sources      = M = 3
Number of destinations = N = 5
Total demand: 21
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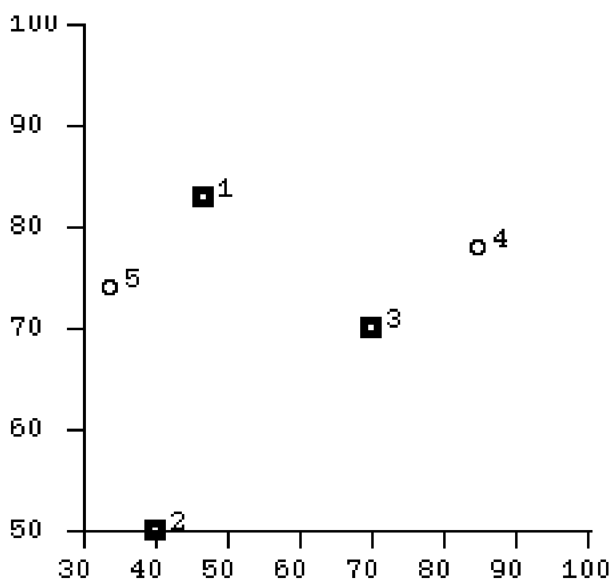
Costs, Supplies, Demands

i/j	1	2	3	4	5	K	F
1	0	30	30	40	20	11	5000
2	30	0	40	50	30	8	2000
3	30	40	0	20	40	9	3000
Demand:	6	2	9	2	2	28	0

K = capacity,
F = fixed cost



a. Formulate the problem as a mixed-integer LP problem:

b. Apply Lagrangian relaxation to the supply constraints of your formulation. Write the Lagrangian subproblem:

c. Letting the Lagrange multipliers be each equal to 10, illustrate how the Lagrangian subproblem is easily solved.

d. What is the objective value of the Lagrangian subproblem? Is it a lower or upper bound on the optimum of the original problem?

e. For each of the Lagrange multipliers, specify whether it should be adjusted upward or downward in order to improve the bound.

6. Cutting Plane Algorithm: Consider the problem

$$\begin{array}{ll} \text{Minimize} & 2X_1 - 0.5X_2 \\ \text{subject to} & 4X_1 - 0.5X_2 \leq 10 \\ & X_2 \geq 2 \\ & X_1, X_2 \geq 0, \ \& \text{ integer} \end{array}$$

After adding slack & surplus variables X_3 & X_4, respectively, and solving the LP relaxation, we get the optimal tableau:

X_1	X_2	X_3	X_4	rhs	
0	0	-0.5	-0.25	4.5	(min)
1	0	-0.125	0.25	2.25	
0	1	-1	0	2	

a. What constraint may be added to the problem to exclude this extreme point of the feasible region of the LP relaxation without excluding any integer feasible solutions?

b. Graph, in (X_1,X_2)-space, the original feasible region of the LP relaxation, and the new feasible region after adding this constraint.

c. Add the new constraint to the tableau, and indicate where the next pivot should be:

X_1	X_2	X_3	X_4	rhs	
0	0	-0.5	-0.25	4.5	(min)
1	0	-0.125	0.25	2.25	
0	1	-1	0	2	

7. Balas' Implicit Enumeration Method. Consider the problem

$$\begin{aligned} \text{Maximize} \quad & -7X_1 + 6X_2 + 4X_3 - 5X_4 - 6X_5 \\ \text{subject to} \quad & -3X_1 - X_2 + 2X_3 + 3X_4 - 3X_5 \le 1 \\ & X_2 - X_3 - 4X_4 - 2X_5 \le -1 \\ & X_1 + 4X_3 + 3X_4 \le 4 \\ & X_i = 0 \text{ or } 1,\ i=1,2,\ldots 5 \end{aligned}$$

a. Convert the problem to the standard form, i.e.,

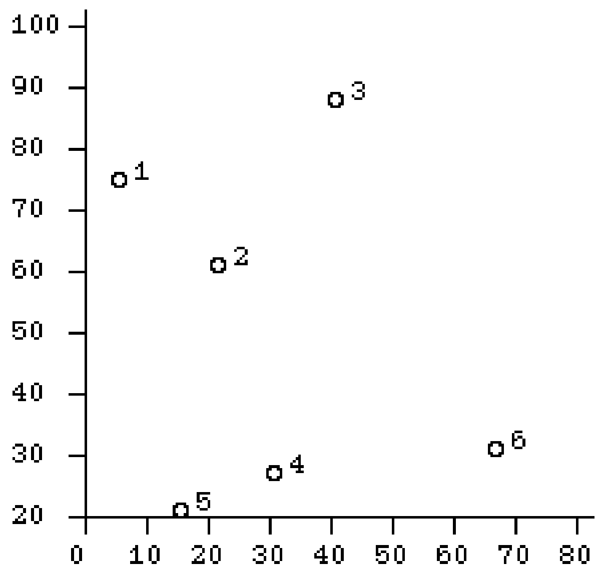
$$\begin{aligned} \text{Minimize} \quad & \Sigma\, c_i x_i \\ \text{subject to} \quad & Ax \le b \\ & X_i = 0 \text{ or } 1 \end{aligned}$$

with $c_i \geq 0$ for all i

b. Suppose you are now at the node represented by J = {$\underline{5}$, -2, $\underline{3}$ } (where indices are of variables in the standard form you found in (a).) Draw the corresponding tree diagram, and indicate this node. What are the free variables? variables fixed at 0? variables fixed at 1? What do the underlines indicate?

c. Try the fathom the current node above by the fathoming tests of Balas' algorithm.

8. Vehicle Routing Problem : Consider the problem below, where a truck depot is located in city #6. Each truck (with capacity 9 units of weight) , and deliveries are to be made to the other five cities, as shown below. (Ignore the delivery to be made in city #6.)



i	1	2	3	4	5	6
X[i]	6	22	41	31	16	67
Y[i]	75	61	88	27	21	31
Wt[i]	2	9	7	3	4	2

Vehicle capacity = 9, with depot at node 6
Total weight of nodes (excluding depot): 25
Average weight per node: 5

Distances

from \ to	1	2	3	4	5	6
1	0	21	37	54	55	75
2	21	0	33	35	40	54
3	37	33	0	62	72	63
4	54	35	62	0	16	36
5	55	40	72	16	0	52
6	75	54	63	36	52	0

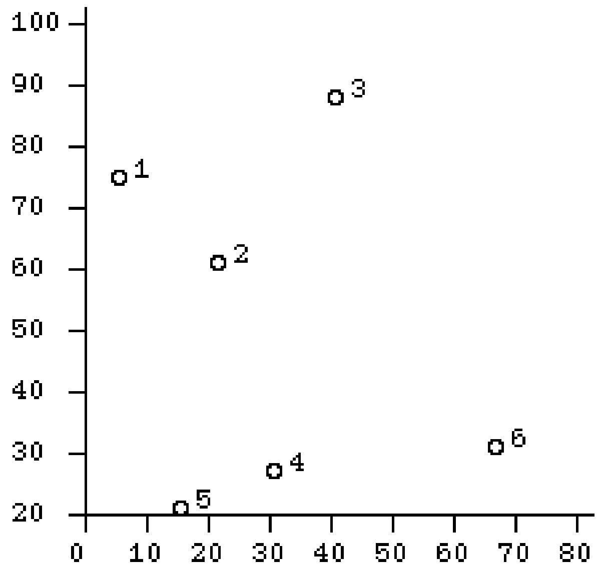
a. Choose a non-zero entry of the savings matrix below and explain how it is computed:

Savings Matrix

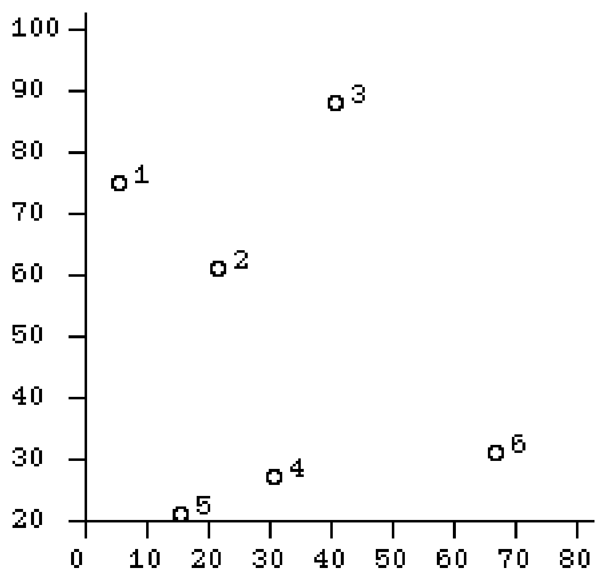
from \ to	1	2	3	4	5	6
1	0	108	101	57	72	0
2	108	0	84	55	66	0
3	101	84	0	37	43	0
4	57	55	37	0	72	0
5	72	66	43	72	0	0
6	0	0	0	0	0	0

Next you will perform the computations of the Clark-Wright heuristic to find a set of delivery routes.

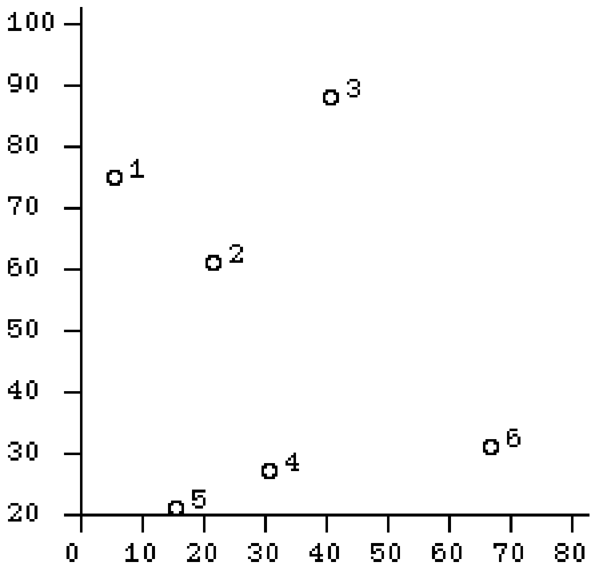
b. Indicate the result of the first step below:



c. Indicate the result of the second step below:



d. Do a third step, and indicate the results below:



9. Benders' Decomposition Algorithm : Consider the following mixed-integer LP problem:

$$\begin{array}{ll} \text{Minimize} & 5x + 4y \\ \text{subject to} & 2x + 3y \geq 7 \\ & 4x + \ \ y \geq 5 \\ & x \geq 0,\ \ y \in \{0,1,2,3\} \end{array}$$

a. Write the LP (primal subproblem) obtained by fixing y, then write the dual subproblem. Let the value of this LP be denoted by v(y).

b. Evaluate v(1), i.e., evaluate the initial trial solution y'=1.

c. Using the result of (b), find a linear approximation $\underline{v}_1(y)$ to the function v(y). Plot this function below:

d. What is the value of $\underline{v}_1(1)$?

e. Minimize $\underline{v}_1(y)$ s.t. $y \in \{0,1,2,3\}$. Call your solution y".

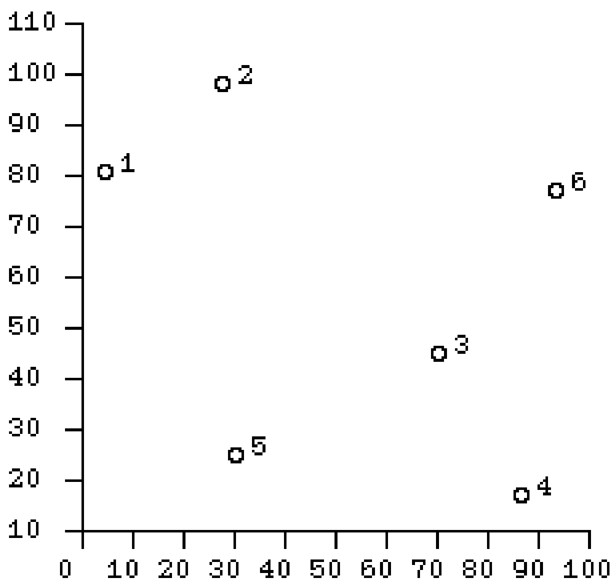
f. What is the value of $\underline{v}_1(y")$?

g. Solve the subproblem again to evaluate v(y").

h. From the result of (g), find an improved approximation $\underline{v}_2(y)$ to v(y). Plot it below:

i. Explain how one should proceed to complete Benders' algorithm. When does one stop?

10. Traveling Salesman Problem : Consider the following traveling salesman problem, where the distances are symmetric.



i	1	2	3	4	5	6
X[i]	5	28	71	87	31	94
Y[i]	81	98	45	17	25	77

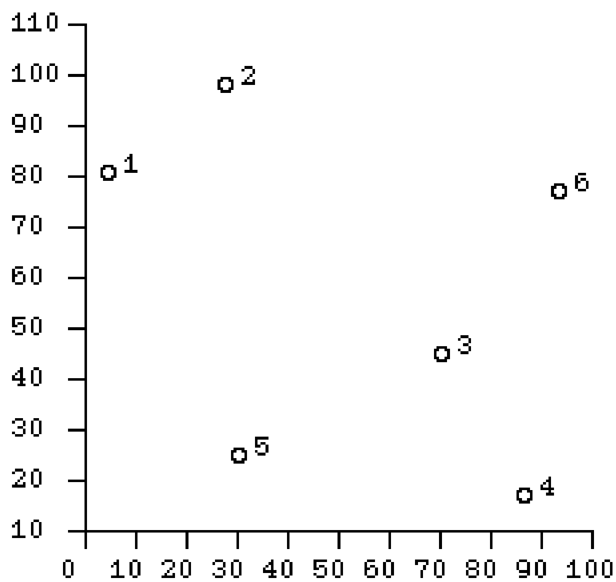
Distances

from \ to	1	2	3	4	5	6
1	0	29	75	104	62	89
2	29	0	68	100	73	69
3	75	68	0	32	45	39
4	104	100	32	0	57	60
5	62	73	45	57	0	82
6	89	69	39	60	82	0

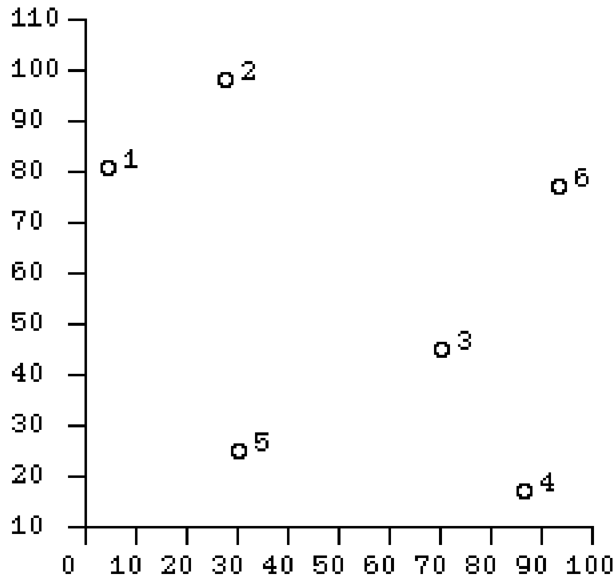
a. Find a minimum spanning 1-tree of these six cities (adding city #1 last). (Indicate the 1-tree on the plot above.)

b. What is the total length of this 1-tree? Does it give an upper or a lower bound on the optimal tour length?

c. Perform one iteration of the vertex penalty method, using a "step size" of 10. Is the new 1-tree a tour? (Indicate the 1-tree below:)



d. Perform the first three iterations of the "farthest insertion" algorithm, so as to obtain a tour of four cities. Indicate that tour below:


110
100
90
80
70
60
50
40
30
20
10
0
10
20
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100
1
2
3
4
5
6