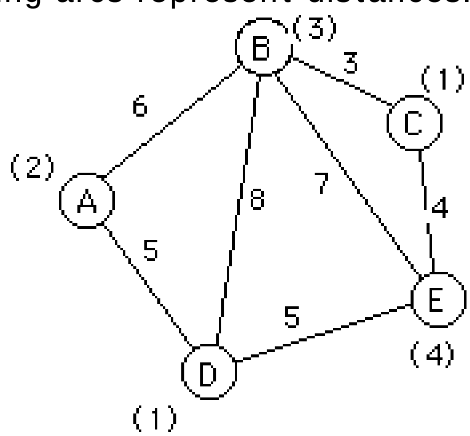
56:272 Integer Pgmg. & Network Flows
Final Examination: 15 December 1988

PART ONE

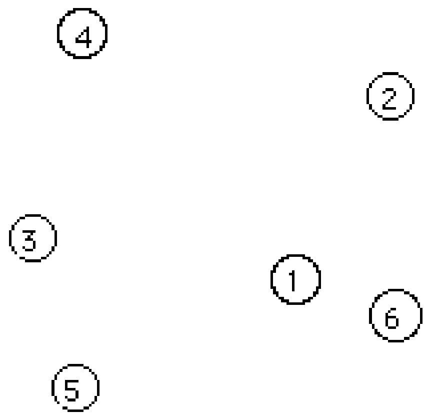
Select ONE of the TWO problems in Part One:

(1.) Facility Location Problems. Consider the network of five cities below, where the numbers in parenthesis represent total demands for a product in those cities, and the numbers on the connecting arcs represent distances.



a. Explain how Dijkstra's labeling algorithm is used to find the shortest distance from a city (e.g. A) to each of the other cities. Do exactly 2 iterations of this algorithm, and indicate the labels of each node at this stage of the algorithm.
b. By inspection, write down the matrix with shortest distances between each pair of nodes.
c. Define what is meant by median of a network. How does a median differ from a center of a network?
d. Find the median of the network.
e. What is meant by a 2-median of a network?
f. Formulate the problem of finding the optimal 2-median set as a (zero-one) integer programming problem, i.e., in which of the five cities should a firm place two warehouses to serve the customers at least cost, assuming distribution costs are proportional to both distance and quantity supplied.
g. Formulate the problem of finding the optimal 2-center set as a (zero-one) integer programming problem, i.e. in which two of the five cities should fire stations be placed, and which of the cities should each fire station serve, so as to minimize the farthest distance which any fire station must travel. (Disregard the demands indicated on the network.)

(2.) Routing Problems: Consider the set of six cities and the symmetric distance matrix below:



i \ j:	1	2	3	4	5	6
1 ÷	0	4	5	8	6	2
2 ÷	4	0	7	3	9	5
3 ÷	5	7	0	4	2	6
4 ÷	8	3	4	0	5	9
5 ÷	6	9	2	5	0	7
6 ÷	2	5	6	9	7	0

a. Use the nearest neighbor heuristic to find a tour.
b. Use the farthest insertion heuristic to find a tour.
c. Find a minimum spanning tree of the set of cities.
d. Find a minimum spanning one-tree of the set of cities. Is it a tour?
e. Assign vertex penalties and perform one iteration of the vertex penalty method. Does it result in a tour?
f. Explain how the vertex penalty method may be interpreted as a Lagrangian relaxation method for the traveling salesman problem. What constraints are being "relaxed"? What is the objective function of the Lagrangian relaxation?
g. Based upon your answers in parts (a) through (f), state an upper and a lower bound on the length of the optimal tour.
h. Explain the reason that the assignment algorithm might be applied to the matrix above. What modification must first be made to the matrix? Does the solution of the assignment problem always satisfy the constraint relaxed in (f)?
i. If node #1 is a vehicle depot, compute the element S_{23} of the "savings" matrix used by the Clark-Wright heuristic. Explain the meaning of this number.

PART TWO

Select THREE of the FOUR problems in Part Two:

(1). Formulation of Plant Location Problem: Four possible locations of plants are being considered, to supply demand of customers in nine cities. One or more plant locations are to be selected. Let

D_j = annual demand of city j (j=1,2,... 9)

K_i = annual production capacity of plant at location i, if built (i=1,2,3,4)

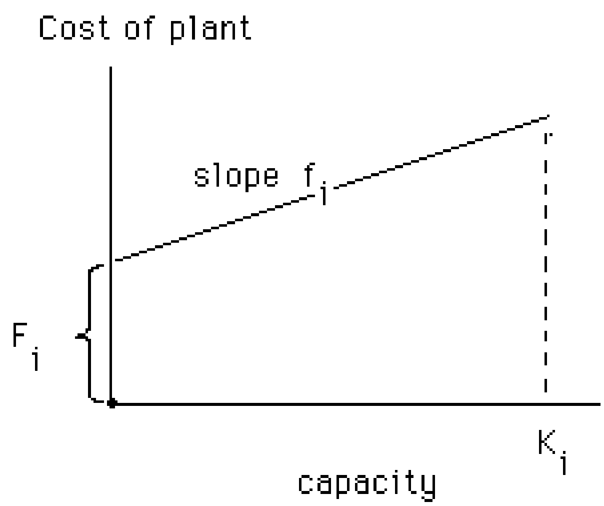
c_{ij} = shipping cost (per unit shipped) between plant i and city j

F_i = annual cost of capital to build & operate a plant at location i

a. Formulate a mixed-integer linear model to choose the locations where plants should be built, using a binary variable Y_i to indicate selection of plant location i. How many integer variables & continuous variables are required? How many constraints are required?

b. Write constraints for each of the additional restrictions:
   i.) Plant 3 should not be built unless either plant 1 or plant 2 is also built.
   ii.) No more than 3 plants should be built.

c. Reformulate the model in part (a) so that the capacity of a plant at location i may be any value between zero and K_i (inclusive), and that the annual cost to build & operate the plant depends upon the capacity which is selected, as indicated in the graph:



d. Reformulate the model in part (c) so that each plant i should not be built unless it is to produce & ship <u>at least</u> a minimum quantity L_i ($0 < L_i < K_i$)

(2.) Benders' Decomposition Algorithm for Plant Location :
Consider the following randomly-generated problem in which demand in 8 cities is to be satisfied by building plants in one or more of four of the cities:

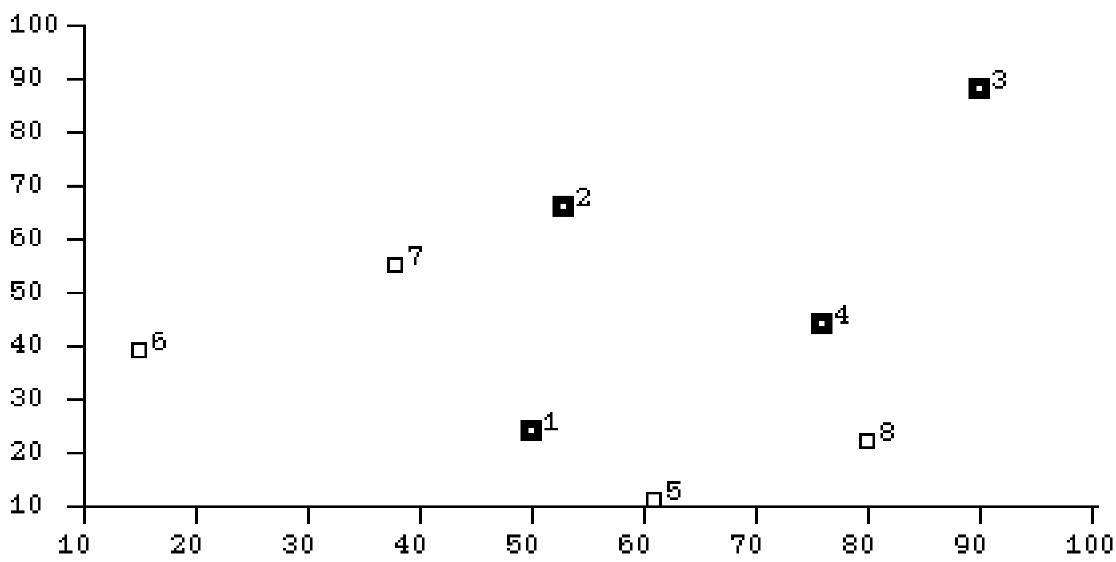
Random Problem (Seed = 3723316)

Number of sources = M = 4
Number of destinations = N = 8
Total demand: 40

Costs, Supplies, Demands

i/j	1	2	3	4	5	6	7	8	K	F
1	0	42	75	33	17	38	33	30	19	498
2	42	0	43	32	56	47	19	52	15	23
3	75	43	0	46	82	90	62	67	17	89
4	33	32	46	0	36	61	40	22	17	129
Demand:	6	1	2	5	5	8	3	10	68	0

K = capacity,
F = fixed cost
<READY?>



Plant Location Problem

(a.) State the mixed-integer programming formulation of the problem. How many continuous variables (X) and how many binary (zero-one) variables (Y) are required?

(b.) Give the expression for the optimal value as a function of Y, i.e. v(Y)

A trial solution was evaluated, in which all four plants are to be open. The result was:

```
                            |------------------------|
                            |      Solution of       |
                            | Transportation Problem |
                            |________________________|
Please enter plant sites to be open
□:
      1 2 3 4
Total capacity exceeds total demand.  Dummy demand point added.
... solving transportation problem ...

Minimum transport cost = 666
Fixed cost of plants = 739
       Total = 1405
CPU time = 9.6 sec.
Generated support αY+b, where α = 498 23 89 129, b = 666
This is support # 1

*** New incumbent! *** (replaces 10000000000)
```

Optimal Shipments

from \ to	1	2	3	4	5	6	7	8	9
1	6	0	0	0	5	8	0	0	0
2	0	1	0	0	0	0	3	0	11
3	0	0	2	0	0	0	0	0	15
4	0	0	0	5	0	0	0	10	2

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(Demand pt #9 is dummy demand for excess capacity.)

NOTE:  Solution is degenerate!
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|--------------------|
|  Dual Solution     |
| of Transportation  |
|     Problem        |
|____________________|

Supply constraints: U = 0 0 0 0 0
Demand constraints: V = 0 0 0 0 17 38 19 22 0

              Reduced costs: COST - U°.+V
            ------------------------------

               0  42  75  33   0   0  14   8
              42   0  43  32  39   9   0  30
              75  43   0  46  65  52  43  45
              33  32  46   0  19  23  21   0
```

(c.) Explain how the "linear support" coefficients α and b above were determined.

(d.) Explain briefly the purpose of the "Master problem":

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                    Master Problem
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Trial set of plants: 1 2 3
with estimated cost 1276 < incumbent ( = 1405)

Current status vectors for Balas' additive algorithm:
    j:      1  3  2 ¯4
underline:  0  0  0  0

CPU time: 1.5 sec.
```

Using this solution of the master problem (which was sub-optimized), the subproblem, i.e. transportation problem, was next solved:

```
|------------------------|
|     Solution of        |
| Transportation Problem |
|________________________|
Please enter plant sites to be open
⎕:
      1 2 3
Total capacity exceeds total demand.  Dummy demand point added.
... solving transportation problem ...

Minimum transport cost = 1078
Fixed cost of plants = 610
       Total = 1688
CPU time = 20.9 sec.
Generated support αY+b, where α = 688 503 871 129, b = ¯374
This is support # 2
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Next the master problem is sub-optimized again:

```
                              Master Problem
                              --------------

Trial set of plants: 1 3 4
with estimated cost 1382 < incumbent ( = 1405)

Current status vectors for Balas' additive algorithm:
    j:      1  3 ¯2  4
underline:  0  0  1  0
```

(e.) State the current approximation $\underline{v}_2(Y)$ of the optimal value function, and illustrate how it is evaluated, using the solution of the master problem above (i.e. plants #1,2,&3 open). That is, how is the value 1382 computed?

(f.) Sketch the enumeration tree, showing the node corresponding to the solution of the master problem above, and indicate parts of the tree which have been fathomed.

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                         |-----------------------|
                         |     Solution of       |
                         | Transportation Problem|
                         |_______________________|
Please enter plant sites to be open
□:
      1 3 4
Total capacity exceeds total demand.  Dummy demand point added.
... solving transportation problem ...

Minimum transport cost = 794
Fixed cost of plants = 716
       Total = 1510
CPU time = 13.05 sec.
Generated support αY+b, where α = 536 23 820 486, b = ¯332
This is support # 3
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(g.) Evaluate $\underline{v}_3(Y)$ for Y representing the opening of plants #2,3,&4. What does this tell you concerning the cost of such a decision? Could this set of plants possibly be optimal?

(3.) Knapsack Problem : A "knapsack" is to be filled to maximize the value of the contents, subject to a weight restriction:

Item	Value($)	Weight (lb.)	Volume(ft^3)
1	10	6	3
2	8	5	2
3	6	3	2
4	3	2	1
5	2	1	2

At most one unit of an item should be included. The total weight of the knapsack cannot exceed 14 lb.

a. Formulate this problem as a 0-1 ILP problem.

b. Use the branch-&-bound algorithm to solve this problem. You may stop after evaluating 5 nodes, if you wish, without guaranteeing optimality. (In this case, what is the incumbent, and what is your best <u>upper</u> and <u>lower</u> bounds on the optimum?)

c. Suppose that we wish to solve this problem, using dynamic programming . What does the state variable represent? Demonstrate how the optimal solution is determined by consulting the attached computer output.

d. Next a volume restriction is also imposed: the total volume of the knapsack cannot exceed 6 ft^3. Suppose we use Lagrangian relaxation, relaxing the volume restriction with a Lagrange multiplier equal to 1. What is the objective function of the Lagrangian relaxation?

e. Consult the attached DP output. Is the solution of the Lagrangian relaxation feasible in the problem with both weight & volume restrictions?

f. Does this provide an upper or lower bound on the optimal solution? What is the value of this bound?

g. Give an economic interpretation of the Lagrange multiplier above. Should it be increased or decreased in the next Lagrangian relaxation?

(4.) Gomory's Fractional Cutting-Plane Algorithm:

Consider the problem:

$$\begin{aligned} \text{Maximize} \quad & 3X_1 + 3X_2 \\ \text{subject to} \quad & 2X_1 + 4X_2 \le 10 \\ & 5X_1 + 4X_2 \le 12 \\ & X_1, X_2 \ge 0 \text{ \& integer} \end{aligned}$$

After adding slack variables X_3 and X_4 and solving the LP relaxation, we get the optimal tableau:

Current LP Tableau

z	1	2	3	4	B
1	0	0	¯0.25	¯0.5	¯8.5
0	0	1	0.417	¯0.167	2.17
0	1	0	¯0.333	0.333	0.667

Variables:
(Negative of) objective function value: z
Original structural variables: 1 2
Original slack variables: 3 4

a. What constraint may be added to the problem to exclude this optimal LP solution without excluding any integer feasible solution?

b. Express the constraint in (a) in terms of the original variables X_1 & X_2.

c. Append the new constraint to the tableau above, and indicate where the next pivot should be. (You need not perform the pivot!)

d. Before adding the constraint, is the tableau "primal feasible"? "dual feasible"?

e. After adding the new constraint, is the tableau "primal feasible"? "dual feasible"?

f. Why is the dual, rather than the primal (i.e., the ordinary) simplex method used to re-optimize after adding the new constraint?