56:272 INTEGER PGMG. & NETWORK FLOWS
Final Examination: 13 December '87

PART ONE

Indicate whether TRUE or FALSE by indicating + and 0, respectively.

___ a. *In a transportation problem, if total suply exceeds total demand, a "dummy" destination must be added before solving.*
___ b. *In a transportation problem with m sources and n destinations, the number of basic variables is $m + n + 1$.*
___ c. *Unless the current basis is degenerate, the dual variables in a transportation problem are uniquely determined.*
___ d. *The dual variables in the transportation problem are used to determine the variable to leave the basis.*
___ e. *The substitution rates of a nonbasic variable for a basic variable in the transportation problem (if one were to compute them) are either 0, +1, or -1.*
___ f. *The dual variables computed during the solution of a transportation problem are always nonnegative.*
___ g. *The "Northwest Corner" method ignores to cost of shipping goods to their destination when making allocations to get an initial feasible solution to a transportation problem.*
___ h. *A minimal spanning tree of a graph is a spanning tree with the smallest number of branches.*
___ i. *A spanning tree must include all nodes of a graph.*
___ j. *An assignment problem may be considered to be a special case of a transportation problem, with all "transportation costs" equal to 1.*
___ k. *The length of an optimal traveling salesman tour is never greater than the length of an optimal tour for the postman problem.*
___ l. *When using branch-&-bound to minimize an integer linear programming problem, the LP relaxation gives us a lower bound on the integer solutions of a subproblem.*
___ m. *In the traveling salesman problem, one must find a tour through all nodes of a graph, while in the postman problem, one must find a path through all arcs (i.e. edges) of a graph.*
___ n. *An ILP (integer linear programming) problem has an unbounded optimum if its LP relaxation has an unbounded optimum.*
___ o. *If x_1 and x_2 are both binary (i.e. 0,1) variables, then $x_1 \leq x_2$ is equivalent to the restriction "If $x_1 = 1$, then $x_2 = 1$."*
___ p. *If Y is a binary variable, then $5-MY \leq X_1 + X_2 \leq 10 + M(1-Y)$ is equivalent to the restriction "If Y=1, then $X_1 + X_2 \geq 5$; otherwise $X_1 + X_2 \leq 10$."*
___ q. *The adjacency matrix of a graph has one row per vertex (node) and one column per arc (edge).*
___ r. *The adjacency matrix of an undirected graph contains only 1's and 0's.*
___ s. *The adjacency matrix of a directed graph contains only 0, +1, and -1's.*
___ t. *A graph may have several spanning trees.*
___ u. *A spanning tree of a graph contains no cycles.*
___ v. *An Euler tour and an Euler path are the same thing.*
___ w. *The adjacency matrix times its transpose is called the "reachability" matrix.*

___ *x. In a maximization ILP problem, the value of the optimal solution is always less than or equal to the value of the optimum of its LP relaxation.*

___ *y. If an ILP problem has no feasible solution, then its LP relaxation is also infeasible.*

___ *z. Given the same cost (or distance) matrix, the value of the optimal solution to the assignment problem is at least as large as the optimal length of the traveling salesman tour.*

PART TWO

SELECT ONE OF THE TWO PROBLEMS IN PART TWO.

1. TRANSPORTATION PROBLEM
Consider the transportation problem below.

FROM \ TO:	D	E	F	G	supply
A	3	7	6	4	5
B	2	4	3	2	2
C	4	3	8	5	3
demand:	3	3	2	2	

a. Use the "Northwest Corner Rule" to find an initial feasible solution.
b. What is the cost of this initial solution?
c. Compute the dual variables.
d. Show how to compute the reduced costs of shipments from B to D (X_{BD}) and C to E (X_{CE}).
e. Which shipment in part (d) will lower the total transportation costs?
f. What amount may be shipped along the route you selected in part (e)?
g. In this new solution, what shipments (give amounts and destinations) are sent from B?
h. By what amount has the transportation cost been lowered?
i. In this new solution, ___ shipments are positive, and ___ are zero, but basic. This is called a _____________ solution.

2. TRAVELING SALESMAN PROBLEM
Five products are to be manufactured weekly on the same machine. The table below gives the cost of switching the machine from one product to another product.

	to:		A	B	C	D	E
from:	A	-	1	8	3	4	
	B	1	-	8	2	3	
	C	1	3	-	5	1	
	D	2	5	6	-	5	
	E	5	3	7	6	-	

After applying the "Hungarian Method" to the above matrix to solve the associated assignment problem (with large number, M, inserted along the diagonal), we have:

from: \ to:	A	B	C	D	E
A	M	0	3	1	3
B	0	M	3	0	2
C	0	2	M	3	0
D	0	3	0	M	3
E	2	0	0	2	M

a. What is the solution of the assignment problem?
b. What is its cost?
c. Is it a valid product sequence?
d. Apply the "Nearest Neighbor" heuristic method to the original cost matrix. What is the total cost of the resulting product sequence?
e. Apply the "Nearest Neighbor" heuristic method to the reduced cost matrix above. What is the total cost of the resulting product sequence?
f. State both best upper and lower bounds on the cost of the optimal sequence.
g. Explain how you would "branch" in order to find the optimal sequence. Select one of the resulting subproblems, and solve the associated assignment problem. Does the solution represent a valid product sequence?
h. Can you now revise either the best upper or lower bound specified in (f)?

PART THREE

SELECT ANY THREE OF THE SIX PROBLEMS FROM PART THREE.

1. FORMULATION USING BINARY VARIABLES .

A company receives an order for four products, which must be manufactured during the next two weeks, for delivery at the end of the second week. Setups of at most three products may be manufactured during a week. Production capacity is 400 units per week. During any week that a product is manufactured, there is a setup cost, and a minimum lot size . Furthermore, products A and B cannot be manufactured during the same week, while product D should not be manufactured in an earlier week than product C. Relevant data is:

Product	Setup Cost Week 1	Setup Cost Week 2	Production Cost $/unit	Minimum Lot Size	Inventory Cost */unit	Quantity Required
A	100	80	1	50	1	100
B	90	100	2	50	2	200
C	50	75	1	25	1	100
D	100	120	2	50	1	300

Formulate an ILP model to find the least-cost production plan to fill this order. How many integer variables are required? How many continuous variables are required?

2. GOMORY'S FRACTIONAL CUTTING-PLANE ALGORITHM

Consider the problem:

$$\text{Maximize } 3X_1 + 3X_2$$
$$\text{subject to } 2X_1 + 4X_2 \le 10$$
$$5X_1 + 4X_2 \le 12$$
$$X_1, X_2 \ge 0 \text{ \& integer}$$

After adding slack variables X_3 and X_4 and solving the LP relaxation, we get the optimal tableau:

Current LP Tableau

z	1	2	3	4	B
1	0	0	¯0.25	¯0.5	¯8.5
0	0	1	0.417	¯0.167	2.17
0	1	0	¯0.333	0.333	0.667

Variables:
(Negative of) objective function value: z
Original structural variables: 1 2
Original slack variables: 3 4

a. What constraint may be added to the problem to exclude this optimal LP solution without excluding any integer feasible solution?
b. Express the constraint in (a) in terms of the original variables X_1 & X_2.
c. Append the new constraint to the tableau above, and indicate where the next pivot should be. (You need not perform the pivot!)
d. Why is the dual, rather than the primal (i.e., the ordinary) simplex method used to re-optimize after adding the new constraint?

3. BALAS' IMPLICIT ENUMERATION ALGORITHM
Consider the problem

$$\text{Maximize } -2X_1 - X_2 + 3X_3 + 2X_4 + 2X_5$$
$$\text{subject to } -3X_1 - X_2 + 2X_4 + X_5 \leq 0$$
$$5X_1 + 5X_2 - 4X_3 + 3X_4 - 2X_5 \geq 5$$

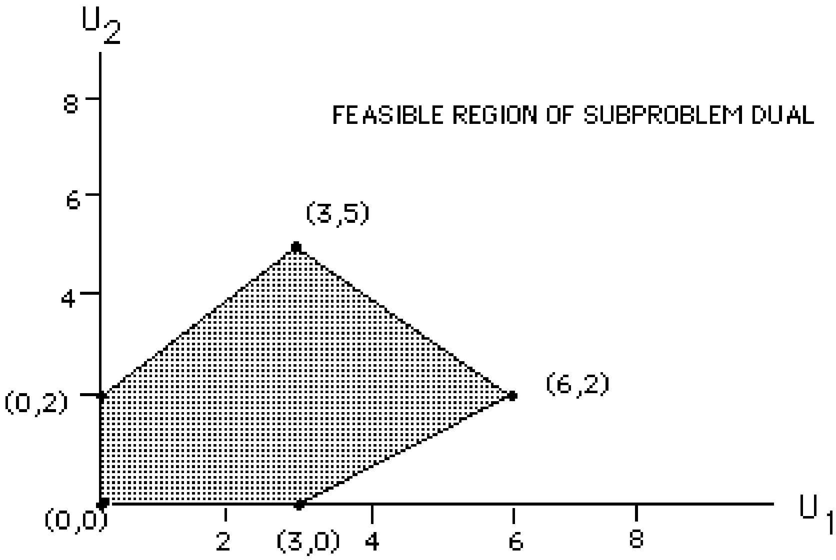
a. Convert the problem to the standard form,i.e., Minimize CX s.t. $AX \leq b$, $X \in \{0,1\}$.
b. Suppose that you are solving the problem (reformulated in part a) and are at the partial solution represented by $J = \{ -\underline{3}, 5, -\underline{2} \}$. Draw the corresponding tree diagram and indicate this node. Which variables are free? ... fixed at value 1? ... fixed at value 0? Indicate which portion of the enumeration tree has been previously fathomed.
c. Is the "zero-completion" of the partial solution in (b) feasible? Why do no other completions of this partial solution need to be considered?
d. Suppose that the node of part (b) is fathomed. What is the next node to be considered?

4. BENDERS' DECOMPOSITION ALGORITHM
Consider the mixed-integer LP:

$$\text{Minimize } 2X_1 + 6X_2 + 8X_3 + 2Y_1 + 3Y_2 + 4Y_3$$
$$\text{subject to } -X_1 + 2X_2 + X_3 + 3Y_1 - Y_2 + 6Y_3 \geq 5$$
$$X_1 - 3X_2 + X_3 + 2Y_1 + 2Y_2 + 4Y_3 \geq 4$$
$$X_j \geq 0, \qquad Y_j \in \{0,1\}.$$

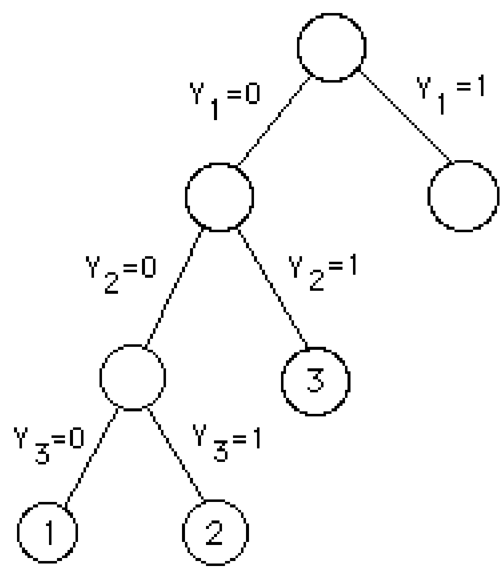
We wish to solve this problem by imbedding Benders' decomposition in an implicit enumeration.

a. Write down the subproblems to be solved, in both the primal and dual form. (The dual feasible region is sketched below.) What is the optimal value function, expressed as a function of Y?



b. Can the primal problem ever be infeasible? Why or why not?

c. Solve the subproblem, starting at node 1 (see tree diagram below) and find your first approximation $\underline{v}_1(Y)$ to the function v(Y). What is the incumbent value (i.e., V^*)?



d. Backtracking from node 1, the next node to be considered is node 2 (see tree diagram). What is the value of $\underline{v}_1(Y)$ at this node? Does the subproblem need to be solved at this node?

e. Suppose, regardless of your answer in (d), that the subproblem is to be solved. What is the new approximation $\underline{v}_2(Y)$?

f. Evaluate $\underline{v}_2(0,1,1)$. Is it < V^*? What can you say about $v(0,1,1)$ relative to $\underline{v}_2(0,1,1)$? Can (0,1,1) be better than the incumbent?

5. KNAPSACK PROBLEM .

A "knapsack" is to be filled to maximize the value of the contents, subject to a weight restriction:

Item	*Value*	*Weight*
1	11	6
2	9	5
3	5	3
4	3	2
5	2	2

At most one unit of an item should be included. The total weight of the knapsack cannot exceed 13.

a. Formulate this problem as a 0-1 ILP problem.

b. Use the branch-&-bound algorithm to solve this problem. You may stop after evaluating 5 nodes, if you wish, without guaranteeing optimality. (In this case, what is the incumbent, and what is your best <u>upper</u> and <u>lower</u> bounds on the optimum?)

c. Suppose that we wish to solve this problem, using dynamic programming . What does the state variable represent? Demonstrate how the optimal solution is determined by consulting the attached computer output.

6. LAGRANGIAN RELAXATION

Recently, a graduate student from the geography department asked me for advice about how to best optimize the following 0-1 Integer LP model (which arises from a problem of selecting wells (i=1,2,...m) to be tested for various ground-water contaminants (j=1,2,...n)):

$$\text{Maximize } \sum_i \sum_j p_{ij}\, x_{ij}$$

$$\text{subject to } \sum_i x_{ij} \geq L_j, \quad j=1, 2, \ldots n$$

$$\sum_i \sum_j C_j\, x_{ij} \leq B \quad \text{(budget limitation)}$$

$$x_{ij} \in \{0,1\}.$$

Discuss how the technique of Lagrangian Relaxation might best be used to solve this problem. In particular, which constraint(s) should be relaxed? Write the subproblem which would need to be solved and discuss a method for solving it. How should the Lagrangian multiplier(s) be adjusted? Illustrate, if possible, using the data below:

	j=	1	2	3	(test)
i=	1)	3	2	5	
(well)	2)	1	5	2	
	3)	5	4	3	(benefit of performing test j
	4)	2	3	4	on well i)
	5)	4	1	2	
	L_j =	2	1	2	(min. # of wells to be tested for j)
	C_j =	2	3	3	(cost of test j)

Budget B = 24

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