A FUZZY WAVE VECTOR DIAGRAM FOR STRONG ACOUSTO-OPTIC INTERACTION

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ABSTRACT

It is shown that for strong acousto-optic interaction, the conventional wave vector diagram must be replaced with a fuzzy triangle representing the dynamics in the interaction region,

INTRODUCTION

It is well known that the wave vector diagram plays a ubiquitous role in the heuristic explanation of acousto-optic interaction [1]. It has been used to explain the Bragg-angle phenomenon, the Doppler shift of the scattered light, the limited bandwidth of modulators and deflectors, the ray tracing formalism for diffracted rays, the non-diffractive nature of Schlieren images of a sound field [2], and the qualitative apects of strong interaction by multiple scattering.

For weak interaction in two dimensions, it is possible to attach a numerical, global description to the wave vector diagram in which the (complex) amplitudes of the various waves involved are connected by a simple product rule: Scattered wave Incident wave x Sound wave. This is no longer the case in strong interaction; a quantitative global formalism in terms of plane waves only is no longer possible. However the scattered spectrum may be derived locally in terms of the incident spectrum and the sound field -- not the sound wave spectrum -- along the so-called Bragg lines [3].

The underlying reason for this failure of the wave vector diagram to give a quantatative global description is that in the interaction region the plane wave amplitude of <u>every</u> order (including the zeroth order) will vary along the nominal propagion direction Z due to a strong exchange of energy. In the interaction region the amplitudes are therefore expressible as Fourier spectra containing terms $\exp(\pm j\kappa z)$where κ is a Z-directed spatial frequency. These terms multiply the basic small-angle propagation term $\exp(-jk_z z)$ to give rise to overall spatial frequencies $k_z \pm \kappa$.

The waves that the terms $\exp(-jk_z z \pm j\kappa z)$ refer to, are, except for k=0, no longer pure plane waves satisfying the wave equation in the unperturbed medium. If we assume that $|k| << k_z$ -- which is true for a relatively slow dependence on z -- these quasi plane waves may be taken into account, at least qualitatively, by making the wave vector diagram fuzzy in the Z- direction, as shown in Fig. 1.

Fig.1a shows the conventional wave vector diagram. Fig.1b symbolically shows the fuzzy diagram that applies to the interaction region. In the fuzzy diagram the rectangular region around the nominal (center) K vector denotes a spectrum of sound vectors of identical length that together compose the sound field. (The top and bottom of the hourglass are straight lines only to a first approximation for $|\kappa| << K$). The spread in k_0 and k_1 represent k vectors of different lengths, composing two spatially varying plane waves with amplitudes $\tilde{E}_0(z)$ and $\tilde{E}_1(z)$. Fig.1b thus represents Bragg

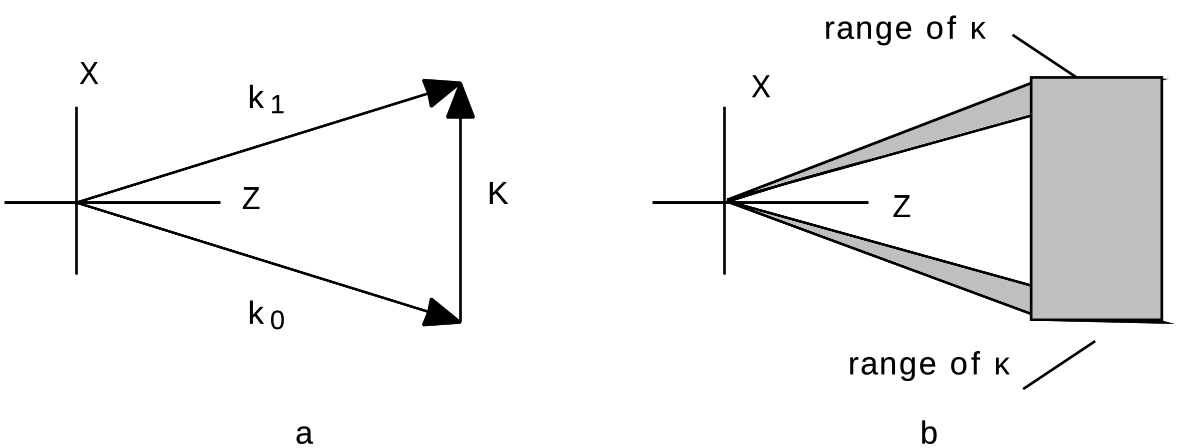

Fig.1

a) Wave vector diagram representing global interaction, b) fuzzy wave vector diagram representing dynamics in the interaction region.

diffraction with an arbitrary sound field that is assumed to be narrow enough not to generate orders other than 0 and 1. Note that the global (overall) representation is still qualitatively given by Fig. 1a, but Fig.1b is representative of the detailed dynamics of the interaction.

In what follows I will show that the fuzzy diagram follows from the conventional strong interaction theory for arbitrary fields. It will be seen that a precise numerical description can be attached to this diagram, that takes the fuzziness into account. Following that, I will apply the formalism to the well known cases of pure Bragg diffraction and near Bragg diffraction. For simplicity I will assume that the sound wave is wide enough for its spectrum to be represented by a delta function. It will be seen that in these two cases the fuzzy wave vector diagram is not smeared out but rather consists of two distinct wave vector triangles.

WEAK INTERACTION

For weak interaction, the situation represented by Fig.1a may be summed up as follows

$$\tilde{E}_1(k_1, z=+\infty) = -ja\, \tilde{S}_1(K)\, \tilde{E}_0(k_0) \qquad (1)$$

where $\tilde{E}_0$, $\tilde{E}_1$, and $\tilde{S}_1$ denote plane wave spectral amplitude densities. The factor a is dependent on the medium and is given by

$$a = -kpn_0^2/4 \qquad (2)$$

where k is the propagation constant in the medium, n_0 its refractive index and p its appropriate elasto-optic coefficient.

Eq. (1) refers to weak interaction, hence

$$\tilde{E}_0(k_0) = \tilde{E}_0(k_0, z=-\infty) = \tilde{E}_0(k_0, z=+\infty) = \tilde{E}_{in}(k_0) \qquad (3)$$

where $\tilde{E}_{in}(k_0)$ denotes the incident plane wave in the spectrum.

STRONG INTERACTION

The general case of upshifted Bragg diffraction with the plane wave of light incident at an angle $\varphi_{inc} < 0$ is shown in Fig. 2. The dotted line labeled 'b' is the Bragg line [3] appropriate to the interaction. It forms the intersection between the zeroth and first orders, and, for small angles φ_{in} and φ_B, is given by

$$x = z(\varphi_{in} + \varphi_B) \tag{4}$$

The interaction between the zeroth and first order is mediated by the sound field along the Bragg line, according to

$$\frac{dE_1}{dz} = -jaS[z(\varphi_{in} + \varphi_B), z]E_0(z) \tag{5}$$

$$\frac{dE_0}{dz} = -jaS^*[z(\varphi_{in} + \varphi_B), z]E_1(z) \tag{6}$$

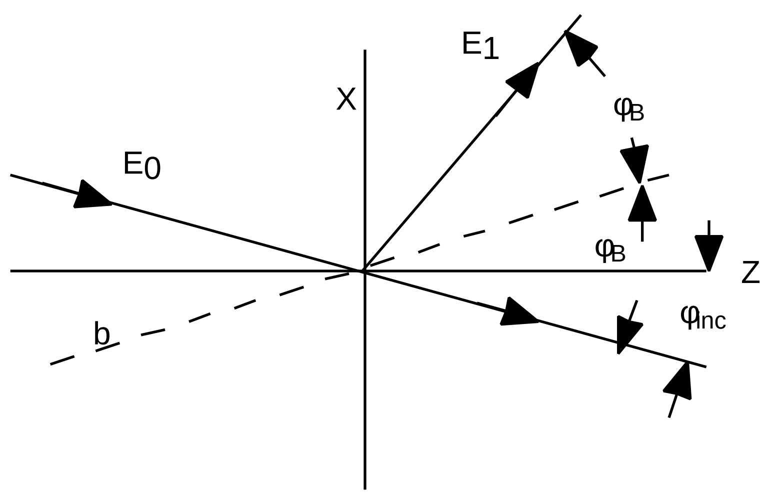

Fig. 2

Interaction diagram illustrating detailed dynamics in the interaction region.

where we have used plane wave amplitudes rather than spectral densities, as only a single plane wave of incident light is assumed.

For simplicity we set $\varphi_{in} = -\varphi_B$ and hence (5) and (6) become

$$\frac{dE_1}{dz} = -jaS[z]E_0(z) \tag{7}$$

$$\frac{dE_0}{dz} = -jaS^*[z]E_1(z) \tag{8}$$

Next we take the Z-directed inverse Fourier transform of both sides of equations (5) and (6), using the conjugate variables z and κ. (The inverse rather than the forward transform is used, because (5) and (6) are predicated on the electrical engineering convention, according to which the plane wave spectrum is the inverse Fourier transform of the spatial distribution.) The inverse Fourier transforms will be indicated by $\hat{E}_0(\kappa)$, $\hat{E}_1(\kappa)$ and $\hat{S}(\kappa)$.

$$-j\kappa\hat{E}_1(\kappa) = -jaF^{-1}[S(z)E_0(z)] = -j\frac{a}{2\pi}\hat{S}(\kappa).*\hat{E}_0(\kappa) \tag{9}$$

$$-j\kappa\hat{E}_0(\kappa) = -jaF[S^*(z)E_1(z)] = -j\frac{a}{2\pi}\hat{S}^*(-\kappa).*\hat{E}_1(\kappa) \tag{10}$$

where F^{-1} denotes the inverse Fourier transform operator, and the symbol * denotes conjugation.

Multiplying (9) by jκ and substituting into (10) we find

$$\kappa^2\hat{E}_1(\kappa) = \frac{a^2}{4\pi^2}\kappa\{\hat{S}(\kappa) * \frac{1}{\kappa}[\hat{S}^*(-\kappa) * \hat{E}_1(\kappa)]\} \tag{11}$$

$$\hat{E}_0(\kappa) = \frac{a}{2\pi\kappa}[\hat{S}^*(-\kappa).*\hat{E}_1(\kappa)] \tag{12}$$

THE FUZZY WAVE VECTOR DIAGRAM

To see how the equations derived correspond to the fuzzy diagram discussed before, we must use explicit expressions for the convolution. From (9) we find

$$\hat{E}_1(\kappa) = \frac{a}{2\pi\kappa}[\hat{S}(\kappa).*\hat{E}_0(\kappa)] = \frac{a}{2\pi\kappa}\int_{-\infty}^{\infty}\hat{S}(\kappa')\hat{E}_0(\kappa-\kappa')d\kappa' \tag{13}$$

An interpretation of (13) is illustrated in Fig. 3

We recognize the weak interaction triangle OAB with the wave vectors k_0, k_1 and K. The value of κ' is set out perpendicular to K along the line BD. The dotted line AE represents a sound wave vector K' in a direction different from K. (As explained before, to a first approximation the lengths of K and K' are identical.) In eq. (9) we interpret $\hat{S}(\kappa')$ as the amplitude (density) of the wave in the direction K'.

Similarly, the point D represents the value κ as indicated in the drawing. Hence $\hat{E}_1(\kappa)$ should be interpreted as the amplitude (density) of the scattered wave in the direction k_1'.Translating the sound wave vector K' parallel to itself to link up with k_1', we find the point C with AC equalling κ - κ' The amplitude density of the corresponding vector k_0' should then be equated with twith the amplitude density $\hat{E}_0$ (κ −κ') from eq. (13).

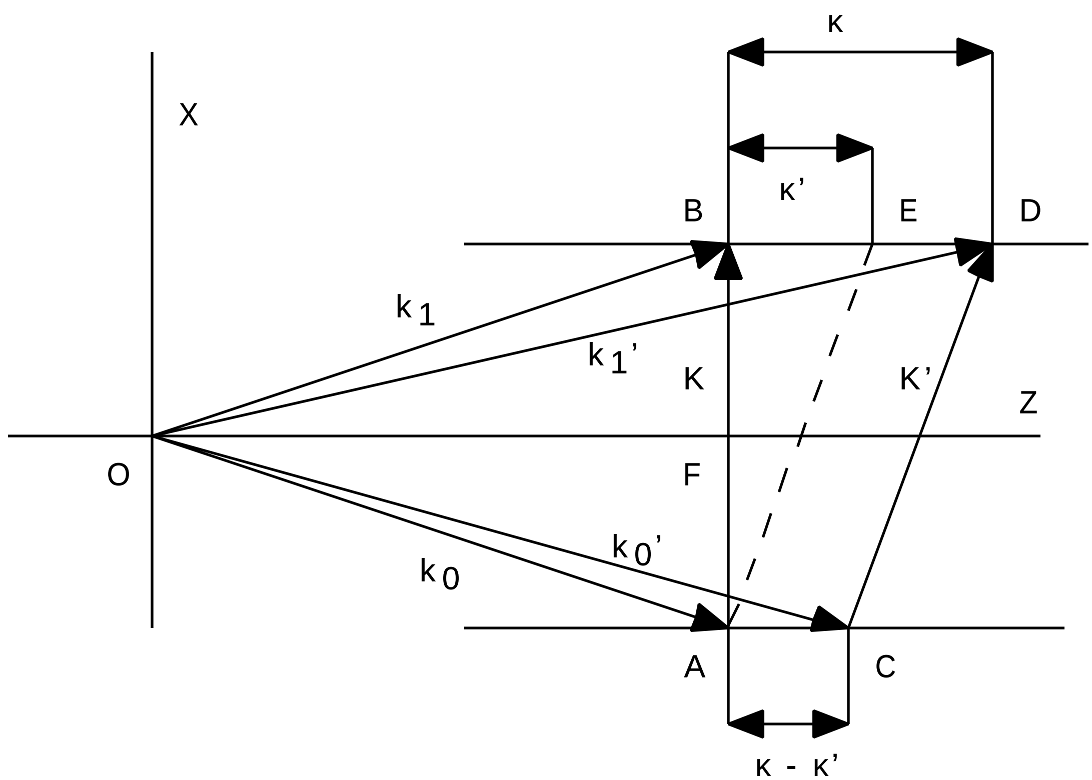

Fig.3

Correspondence of fuzzy wave vector diagram and convolution integral of interaction.

Putting everything together we see that the contribution $\hat{S}(\kappa')\,\hat{E}_0(\kappa-\kappa')d\kappa'$ of the integral in (9) may be interpreted as the elementary interaction represented by triangle OCD with wave vectors k_0', k_1' and K'. In general there are infinitely many such contributions to k_1', mediated by different vectors K'. They are found by varying the angle of K' , corresponding to varying κ' in the integral of eq. (13).

Summing up we conclude that in the general case the fuzzy triangle represents an infinity of interactions, each characterized by a wave vector triangle. The (optical) waves involved are strictly speaking not plane waves ; they have the character of eigen-waves in the perturbed medium. The sound waves are true plane waves; they form the plane wave spectrum of the sound beam.

CONVENTIONAL STRONG BRAGG DIFFRACTION

Let us now consider the case where

$$S(x) = \mathrm{Lim}\{S_0 \mathrm{rect}(2x/L)\} \text{ for } L \rightarrow \infty \tag{14}$$

so that

$$\hat{S}(\kappa) \rightarrow 2\pi S_0 \delta(\kappa) \tag{15}$$

Eq. (11) becomes

$$[\kappa^2 - 4\pi^2 a^2 S_0^2]\hat{E}_1(\kappa) = 0 \tag{16}$$

with the solution

$$\hat{E}_1(\kappa) = A\delta(\kappa - aS_0) + B\delta(\kappa + aS_0) \tag{17}$$

$$E_1(x) = A\exp(-jaS_0 z) + B\exp(jaS_0 z) \tag{18}$$

From (12) we find, with (15),

$$\hat{E}_0(\kappa) = \frac{a}{2\pi\kappa}\ [\hat{S}^*(-\kappa).*\ \hat{E}_1(\kappa)] = \frac{aS_0}{\kappa}\ \hat{E}_1(\kappa) =$$
$$A\delta(\kappa - aS_0) - B\delta(\kappa + -jaS0aS_0) \qquad (19)$$

$$E_0(x) = A\exp(-jaS_0z) - B\exp(jaS_0z) \qquad (20)$$

Imposing on (18) and (20) the boundary conditions

$$E_1(-L/2) = 0,\ E_0(-L/2) = E_{inc.} \qquad (21)$$

we find, after some algebra, the well known result

$$E_1 = -jE_{inc}\sin[aS_0(z+L/2)] \qquad (22)$$

$$E_0 = E_{inc}\cos[aS_0(z+L/2)] \qquad (23)$$

A graphical interpretation of the conventional strong Bragg diffraction case is shown in Fig.4. The conventional wave vector diagram is given by dashed triangle OAB. The more relevant fuzzy wave vector diagram is given by the combination of triangles OCD and OEF. The relevant values of κ are $\pm aS_0$. Note that for $S_0 \to 0$ the fuzzy diagram becomes the conventional one.

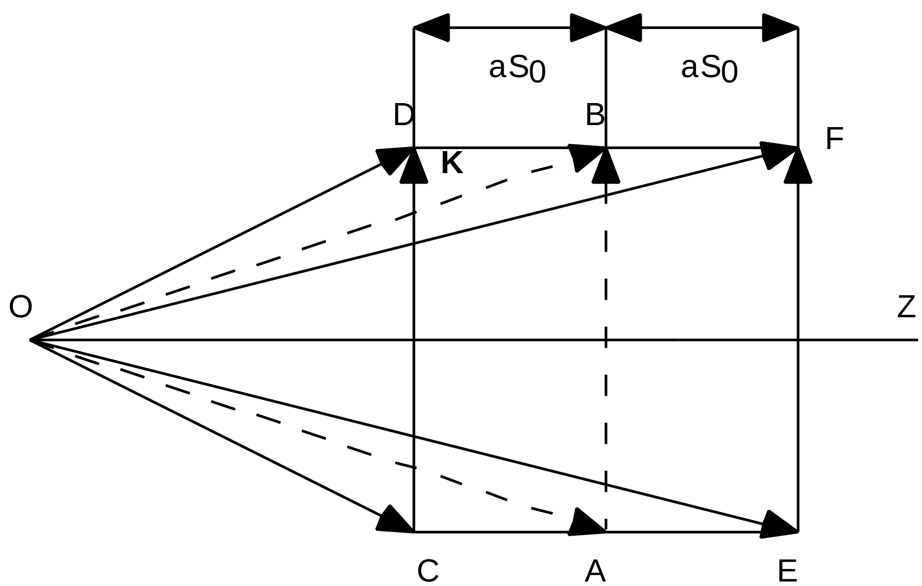

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Fig.4

Fuzzy wave vector diagram (triangles OCD and OEF) for pure Bragg interaction.

THE NEAR-BRAGG REGIME

In this case the nominal direction of the soundwave is no longer at the exact direction required for Bragg diffraction. Nevertheless interaction is still possible. For weak diffraction this is usually explained by considering the plane wave spectrum of the sound beam. For small angles, this spectrum is proportional to sinc($K\theta L/2\pi$), where θ is the propagation angle of the plane wave considered. Thus, the reasoning goes, if the soundbeam is off by an angle β, there is still a plane wave available for interaction, albeit with reduced amplitude sinc($K\beta L/2\pi$). For strong interaction this reasoning breaks down. This has been shown most clearly by Molchanov [4] through what I have called the Molchanov paradox..

Consider a zero in the sound plane wave spectrum at, say, $K\beta L/2\pi = 1$. If the direction of the sound beam is off by that amount, then there will be no plane wave available for interaction. And, indeed, for weak interaction we find that no diffraction takes place. Conventional near-Bragg diffraction theory however indicates that interaction still will take place at that angle for higher sound levels. This in spite of the fact that the plane wave of sound responsible (in the conventional weak interaction explanation) is absent. It is thus clear that a picure based on the conventional wave vector diagram, fails. In what follows I shall work through this example using the fuzzy wave vector diagram.

Let us model the soundbeam by

$$\hat{S}(\kappa) = 2\pi S_0 \delta(\kappa - K\beta) \qquad (24)$$

where $K\beta$ is the offset angle.

Substituting (24) into (11), carrying out the convolutions carefully, including the factor $1/\kappa$ where appropriate, we find

$$\hat{E}_1(\kappa)[\kappa^2 - \kappa K\beta - a^2 S_0^2] = 0 \qquad (25)$$

whence

$$\kappa_1 = \frac{K\beta + \sqrt{K^2\beta^2 + 4a^2S_0{}^2}}{2} \tag{26}$$

$$k_2 = \frac{K\beta - \sqrt{K^2\beta^2 + 4a^2S_0{}^2}}{2} \tag{27}$$

and hence

$$\hat{E}_1(\kappa) = A\delta(\kappa - \kappa_1) + B\delta(\kappa - \kappa_2) \tag{28}$$

With (12) and (24) we find from (28)

$$\hat{E}_0(\kappa) = \frac{aS_0}{\kappa} E_1(\kappa + K\beta) = \frac{aS_0A}{\kappa_1 - K\beta}\delta(\kappa - \kappa_1 + K\beta) + \frac{aS_0B}{\kappa_2 - K\beta}\delta(\kappa - \kappa_2 + K\beta) \tag{29}$$

or

$$E_1(z) = A\exp(-j\kappa_1 z) + B\exp(-j\kappa_2 z) \tag{30}$$

$$E_0(z) = \frac{aS_0A}{\kappa_1 - K\beta}\exp[-j(\kappa_1 - K\beta)z] + \frac{aS_0B}{\kappa_2 - K\beta}\exp[-j(\kappa_2 - K\beta)z] \tag{31}$$

Applying the boundary conditions (21) it is found, after tedious algebra, that

$$E_1(z=L/2) = -jaS_0LE_{in}\exp(j\frac{K\beta L}{2})\mathrm{sinc}(\frac{\sqrt{K^2\beta^2 + 4a^2S_0{}^2}}{2\pi}L) \tag{32}$$

or

$$E_1(z=L/2) = -j\frac{v}{2}E_{in}\exp(j\frac{K\beta L}{2})\,\mathrm{sinc}(\frac{\sqrt{K^2\beta^2L^2 + v^2}}{2\pi}) \tag{33}$$

where we have used the Raman-Nath parameter

$$v = 2aS_0L \tag{34}$$

Eq. (33) throws some light on the inapplicability of the conventional wave vector diagram. Take first the weak interaction case $v \to 0$. We find from (33) that

$$E_1(z=L/2) \to -j\frac{v}{2} E_{in}\exp(j\frac{K\beta L}{2})\,\mathrm{sinc}(\frac{K\beta l}{2\pi}) \text{ for } v \to 0 \qquad (32)$$

As pointed out before, in this regime the interaction is proportional to the amplitude of the plane wave of sound propagating in the direction required by the conventional wave vector diagram. (Relative to the main direction of the sound beam this plane wave propagates in the direction $-\beta$.) The interaction vanishes for an angle β given by $K\beta L = \pm 2\pi$ because the required plane wave is absent. However, (33) shows that when the sound pressure increases, interaction at this angle returns. As I have said before, this is Molchanov's paradox and shows up the insufficiency of the conventional wave vector diagram.

The fuzzy diagram for the near Bragg case analyzed above is illustrated in Fig. 5.

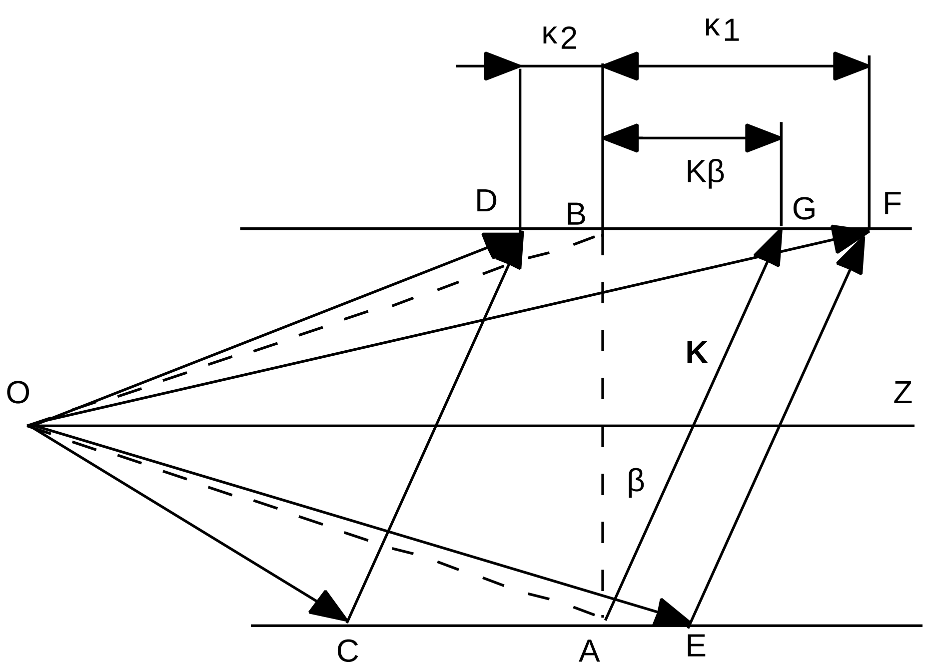

Fig.5

Fuzzy wave vector diagram (triangles OCD and OEF) for near-Bragg interaction

As in Fig. 4, the dashed triangle OAB is the conventional wave vector diagram . The fuzzy diagram consists of triangles OCD and OEF. According to (26) and (27), for $S_0 \to 0$, point F, denoting κ_1, will move toward B and point D, denoting κ_2 , will move toward B. When $\beta=0$ the diagram reverts to the one shown in Fig.4.

DISCUSSION

In the examples given, the fuzzy diagram turns out to consist of two wave vector diagrams. It would have been more convincing to show a continous fuzziness, but this is mathematically more involved , as it would require for example a Gaussian beam input (do-able; a conventional solution is available [5] for guidance) or a continous sound beam spectrum (problematic, although solutions for curved sound beams exist [6,7]). I hope to tackle these two cases in the future.

CONCLUSION

The analysis shows clearly the inapplicability of the conventional wave vector diagram to strong interaction and the necessity to replace it by a fuzzy diagram. Analysis of known cases shows complete agreement with final results. Whether this method of analysis is more practical than conventional ones remains to be seen. It does however offer a new interpretation and as such provides fresh insight into practical problems.

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