6. WAVE EQUATIONS AND WAVES

MAXWELL'S EQUATIONS

An optical field propagating in a homogeneous, isotropic, linear medium, devoid of free charges and curents, must satisfy the following Maxwell's equations:

$$\nabla \times \mathbf{e} = -\frac{\partial \mathbf{b}}{\partial t} \tag{6.1}$$

$$\nabla \times \mathbf{b} = \frac{1}{c^2}\frac{\partial \mathbf{e}}{\partial t} \tag{6.2}$$

$$\nabla \cdot \mathbf{e} = 0 \tag{6.3}$$

$$\nabla \cdot \mathbf{b} = 0 \tag{6.4}$$

where the vectors **e** and **b** are the electric and magnetic field quantities, and c is the light velocity in the medium. In terms of the velocity c_V of light in vacuum

$$c = c_V/n, \tag{6.5}$$

where n is the refractive index of the (nonmagnetic) medium:

$$n = \sqrt{\varepsilon_r}$$

and ε_r the relative dielectric constant

Taking the curl of (6.1) and substituting (6.2) into the result we find

$$\nabla \times \nabla \times \mathbf{e} = \nabla(\nabla \cdot \mathbf{e}) - \ {}^{2}\mathbf{e} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{e}}{\partial t^2} \qquad (6.6)$$

With (6.3) we then find

$$\boxed{{}^{2}\mathbf{e} = \frac{1}{c^2}\frac{\partial^2 \mathbf{e}}{\partial t^2}} \qquad (6.7)$$

with an identical equation for the magnetic field:

$${}^{2}\mathbf{b} = \frac{1}{c^2}\frac{\partial^2 \mathbf{b}}{\partial t^2} \qquad (6.8)$$

Eqs (6.6) and (6.7) are vector wave equations. Similar (scalar) equations must be obeyed by each component of **e** and **b**.

HELMHOLTZ EQUATION

If the field is monochromatic at frequency ω, **e** and **b** are represented by the phasors **A** and **B** :

$$\mathbf{e} = \mathrm{Re}\,\{\mathbf{A}\exp(-j\omega t)\}$$

$$\mathbf{b} = \mathrm{Re}\{\,\mathbf{B}\exp(-j\omega t)\}$$

Maxwell's equations for free space then become

$$\nabla \times \mathbf{E} = j\,\omega\mathbf{B} \qquad (6.9)$$

$$\nabla \times \mathbf{B} = \frac{-1}{c^2}\,j\omega\mathbf{E} \qquad (6.10)$$

$$\nabla \cdot \mathbf{E} = 0 \qquad (6.11)$$

$$\nabla \cdot \mathbf{B} = 0 \qquad (6.12)$$

frequency ω.

Taking the curl (∇ ×) of (6.1), using the relation

$$\nabla \times \nabla \times E = \nabla\ (\nabla . E) - \quad {}^{2}E$$

and substituting into (6.2) we find the well known *Helmholtz equation* :

$$\boxed{{}^{2}\mathbf{E} + k^{2}\mathbf{E} = 0} \qquad (6.13)$$

where the propagation constant k is given by

$$k = \omega / c = 2\pi / \lambda \qquad (6.14)$$

and λ is the wave length in the medium.

SCALAR FIELDS

In this book we will mostly use scalar wave propagation as a model, with the field propagating nominally in the Z direction and E representing a single component in the XZ plane. Eq. (6.6) will then be written as the scalar equation :

$$\boxed{{}^{2}E + k^{2}E = 0} \qquad (6.15)$$

All the following equations in this section will be written as scalar equations.

COMPLEX ENVELOPE EQUATION

In this case e and b are defined as slowly time - varying phasors:

$$e = \mathrm{Re}\ \{E(x,y,z,t)\exp(-j\omega t)\} \qquad (6.16)$$

$$b = \mathrm{Re}\{\, B(x,y,z,t)\exp(-j\omega t)\} \qquad (6.17)$$

where it is assumed that the time variation of A(t) and B(t) is slow compared to ω. Such slowly time-varying phasors are called complex envelopes in communication theory [Ref. 1] Substituting (6.16) into the scalar version of (6.7) we obtain:

$$^{2}E = \frac{1}{c^{2}}\frac{\partial^{2}E}{\partial t^{2}} - \frac{2j\omega}{c^{2}}\frac{\partial E}{\partial t} - k^{2}E \qquad (6.18)$$

Assuming that $\left|\frac{\partial^{2}E}{\partial t^{2}}\right| << \omega \left|\frac{\partial E}{\partial t}\right|$ (6.19)

we may write (6.18) as the *complex envelope equation* :

$$\boxed{\,^{2}E + \frac{2j\omega}{c^{2}}\frac{\partial E}{\partial t} + k^{2}E = 0} \qquad (6.20)$$

COMPLEX PROFILE EQUATION

Often we assume that the field propagates nominally in the Z direction:

$$E(x,y,z,t) = E_e(x,y,z)\exp(jkz) \qquad (6.21)$$

where the variation of the *complex profile* E_e with z is slow compared to k.

In that case

$$\frac{\partial^{2}E}{\partial z^{2}} = \{\frac{\partial^{2}E_e}{\partial z^{2}} + 2jk\frac{\partial E_e}{\partial z} - k^{2}E_e\}\exp(jkz) \qquad (6.22)$$

may be written as

$$\frac{\partial^{2}E}{\partial z^{2}} = (2jk\frac{\partial E_e}{\partial z} - k^{2}E_e)\exp(jkz) \qquad (6.23)$$

if we assume that

$$\left|\frac{\partial^2 E}{\partial z^2}\right| << k \left|\frac{\partial E_e}{\partial z}\right| \qquad (6.24)$$

Substituting (6.23) into (6.15) we find the *complex profile equation*:

$$\boxed{\frac{\partial^2 E_e}{\partial x^2} + \frac{\partial^2 E_e}{\partial y^2} + 2jk\frac{\partial E_e}{\partial z} = 0} \qquad (6.25)$$

COMPLEX ENVELOPE/PROFILE EQUATION

Assuming that $E_e = E_e\,(x,y,z,t)$ is also slowly time varying, it is readily seen that we may "combine" (6.20) and (6.25) to obtain the *complex envelope/profile equation.* :

$$\frac{\partial^2 E_e}{\partial x^2} + \frac{\partial^2 E_e}{\partial y^2} + 2jk\frac{\partial E_e}{\partial z} + \frac{2j\omega}{c^2}\frac{\partial E_e}{\partial t} = 0 \qquad (6.26)$$

which may also be written as

$$\boxed{\frac{\partial^2 E_e}{\partial x^2} + \frac{\partial^2 E_e}{\partial y^2} + 2jk\frac{\partial E_e}{\partial z} + \frac{2jk}{c}\frac{\partial E_e}{\partial t} = 0} \qquad (6.27)$$

PLANE WAVE

A scalar plane wave propagating in the direction of the wave vector **k** is written as

$$\boxed{E(x,y,z) = E_0\exp(j\mathbf{k}.\mathbf{r}) = E_0\exp(jk_x x+jk_y y+jk_z z)} \qquad (6.28)$$

where **r** is the position vector:

$$\mathbf{r} = x\mathbf{a_x} + y\mathbf{a_y} + z\mathbf{a_z} \qquad (6.29)$$

and $\mathbf{a_x}$, $\mathbf{a_y}$, $\mathbf{a_z}$ are unit vectors in the X and Y direction.

The variables k and k_y denote the lengths of the components of the wave vector k in the X and Y direction where

$$k = \omega/c = 2\pi/\lambda = \sqrt{k_x^2 + k_y^2 + k_z^2} \qquad (6.30)$$

Eq. (6.30) follows directly by substituting (6.28) into (6.8). However, strictly speaking, (6.28) does not satisfy eq. (6.3) unless $k_x = k_y = 0$ (see Ex. 6.1). In scalar optics we assume that k_x and k_y are sufficiently small, so that (6.3) is satisfied. In other words, we assume that the wave propagates in a direction not too far off-axis. This is called paraxial propagation.

PARAXIAL APPROXIMATION

Fig. 6.1 shows the angles φ_x, φ_y, and φ_z are the angles that k makes with the X, Y and Z axis respectively. In terms of the direction cosines:

$$k_x = k\cos\varphi_x,\ k_y = k\cos\varphi_y,\ k_z = k\cos\varphi_z \qquad (6.31)$$

It is often more convenient to express this in terms of the azimuth angle φ (angle between k and its projection on the YZ plane) and the elevation angle φ' (angle between k and its projection on the XZ plane), as illustrated in Fig. 6.1

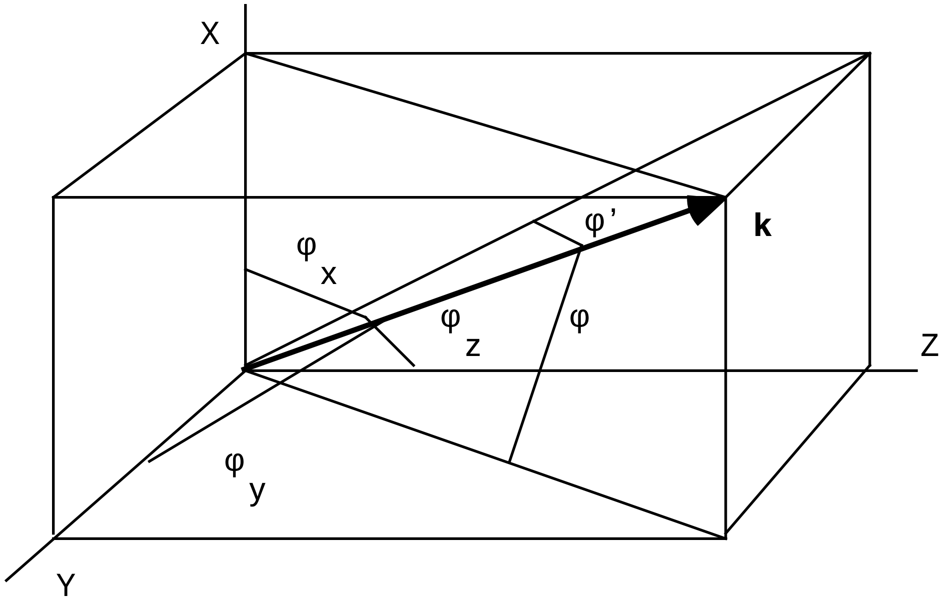

Fig. 6.1

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$$k_x = k\sin\varphi,\ k_y = k\sin\varphi',\ k_z = k\sqrt{1-\sin\varphi^2 - \sin\varphi'^2} \qquad (6.32)$$

In many cases **k** is directed at only a small angle with respect to the Z axis. This is called paraxial propagation and is characterized by $\varphi,\varphi' << 1$ i.e. $\varphi \approx k_x/k$ and $\varphi' \approx k_y/k$ In such cases we may write approximately:

$$k_z = \sqrt{k^2-k_x^2-k_y^2} \approx k-k_x^2/2k-k_y^2/2k \approx k-k\varphi^2/2-k\varphi'^2/2$$

(6.33)

and

$$E(x,y,z) \approx E_0\exp[(jk-k_x^2/2k-k_y^2/2k)z]\ \exp(jk\varphi x + jk\varphi' y) =$$

$$E_0\exp[jk(1-\varphi^2/2 - \varphi'^2/2)z]\exp(jk\varphi x + jk\varphi' y)$$

(6.34)

EVANESCENT WAVE

Note from (6.30) that, if $k_x^2 + k_y^2 > k^2$, the z-component of k i.e. k_z becomes imaginary. The wave then propagates in the +Z direction as exp(kz) or exp (-kz):

$$E(x,y,z) = E_0 \exp(\pm k_z z) \exp(jk_x x + jk_y y) \qquad (6.35)$$

The "plus z" version is obviously physically impossible for all z>0, because of the implied unlimited growth of the wave. The "minus z" version is possible, provided the wave originates from a current or charge carrying surface, located at a finite value of z. This kind of wave is called an evanescent wave and decays quickly away from the surface that originated it.

WAVE FRONTS

A surface on which the phase of the wave is constant is called a wavefront. For the plane wave of (6.28), the wavefronts are obviously given by the surfaces

$$\frac{\mathbf{k}}{k} \cdot \mathbf{r} = d \qquad (6.36)$$

where d is a real constant. As shown in Fig. 6.2, these surfaces are planes perpendicular to **k**, a distance d from the origin.

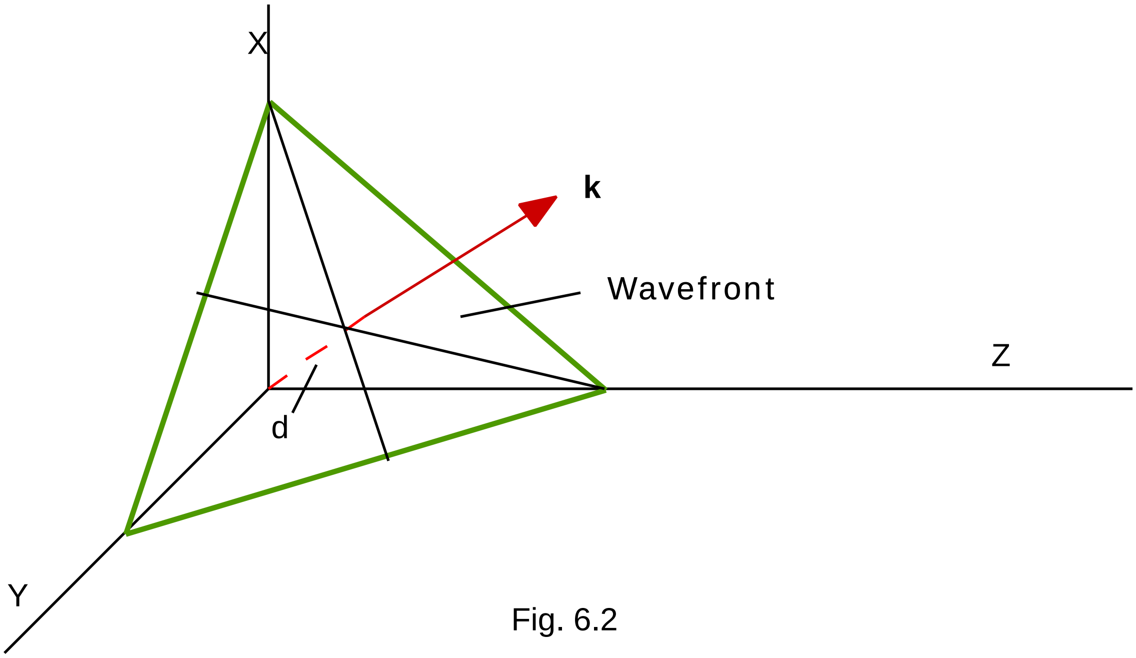

Fig. 6.2

` **SPHERICAL WAVE**

In the case of spherical waves the field is given by

$$E(x,y,z) = \frac{1}{r}\exp(\pm jkr) = \frac{1}{r}\exp(\pm jk\sqrt{x^2+y^2+z^2}) \qquad (6.37)$$

where the plus sign denotes a diverging wave (going out from the source at z=0) , the minus sign a converging wave (coming in to the sink at z=0). As shown in Ex 6.2, expression 6.37 satisfies the Helmholtz equation.

It is readily seen from (6.37) that the wavefronts of a spherical wave are spheres:

$$r = \text{constant} \qquad (6.38)$$

In paraxial propagation the field does not propagate at too steep an angle i.e., it is limited to a region where $x^2 \ll z^2$, $y^2 \ll z^2$, and we may write:

$$E(x,y,z) \approx \frac{1}{z} \exp(\pm jkz) \exp(\pm jkx^2/2z \pm jky^2/2z) \tag{6.39}$$

As before, the plus sign denotes a diverging wave, the minus sign a converging wave. The factor $\exp(\pm jkz)$, being independent of x and y, is frequently left out. The amplitude factor $1/r$ is approximated by $1/z$ while the phase factor (being more sensitive to small changes in r) is approximated to order x^2 and y^2. Note that, in the factor $\exp(\pm jkx^2/2z \pm jky^2/2z)$, z is the radius of the wavefront through the point z. The factor itself is an approximation for the phase in the plane perpendicular to the Z axis at z. In general, factors such as

$$\exp(\pm jkx^2/2R \pm jky^2/2R)$$

denote a spherical wavefront of radius R whereas

$$\exp(\pm jkx^2/2R_1 \pm jky^2/2R_2)$$

denotes a wavefront with radius R_1 in the X direction and radius R_2 in the Y direction. A heuristic explanation of (6.39) - limited to the XZ plane - is shown in Fig. 6.3

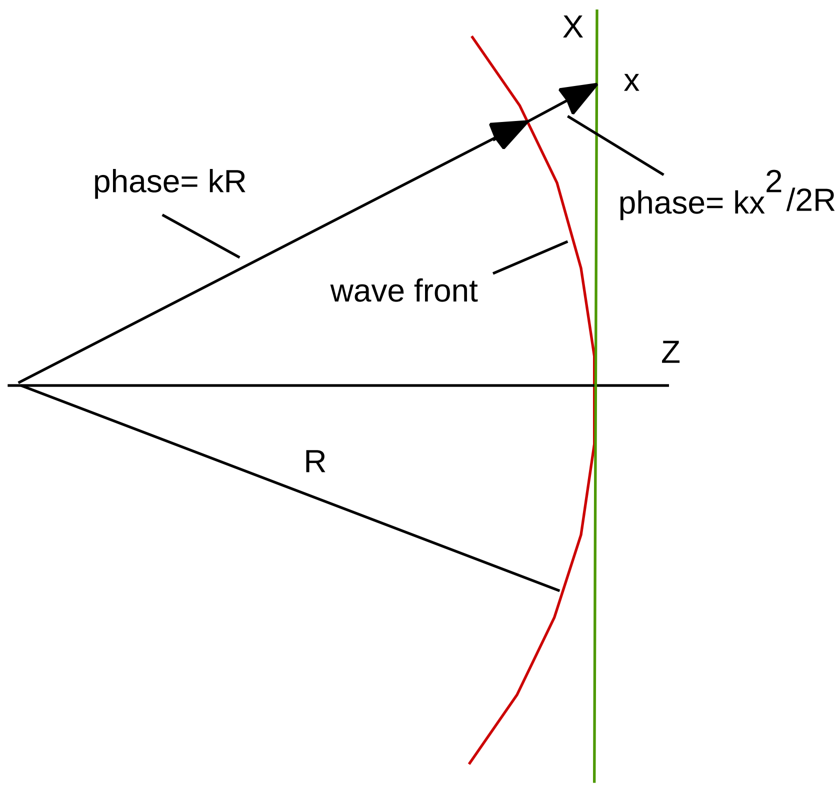

Fig. 6.3

CYLINDRICAL WAVE

A cylindrical wave is essentially a one-dimensional spherical wave, i.e., it is independent of y and varies in the XZ plane only, with $r = \sqrt{x^2+z^2}$. The amplitude is then proportional to $1/\sqrt{r}$ rather than $1/r$, i.e.

$$E(x,y,z) = \frac{1}{\sqrt{r}} \exp\ (\pm jkr) = \frac{1}{\sqrt{r}} \exp(\pm jk\sqrt{x^2+z^2}) \qquad (6.40)$$

As shown in Ex. 6.3, eq.(6.40) satisfies the Helmholtz equation for kr>>1, i.e., r >> λ.

It follows immediately from (6.40) that the wavefronts of a cylindrical wave are cylinders with axes along y:

$$r = \sqrt{x^2+z^2} = \text{constant}. \qquad (6.41)$$

In the paraxial approximation (6.40) becomes:

$$E(x,y,z) \quad \frac{1}{\sqrt{z}} \exp(\pm jkz)\exp(\pm jkx^2/2z) \tag{6.42}$$

CIRCULAR GAUSSIAN BEAM

The field of a circular Gaussian beam at z is given by :

$$E(x,y,z) = E_0 \frac{w_0}{w_z} \exp[\ +jkz-j\eta(z)\ +j\frac{kr^2}{2R_z} + \frac{r^2}{w_z{}^2}\ ] \tag{6.43}$$

where w_0 is the waist (beam radius where the amplitude has decreased to 1/e of the maximum value) at z=0; $w_z = w_z(z)$, the waist at z, and r is a transverse coordinate $r = \sqrt{x^2+y^2}$. The function $\eta(z)$ is a slowly varying phase term. The parameter $R_z = R_z(z)$ is called the radius of curvature of the wavefront at z, in analogy to the paraxial approximation (6.37) of the spherical wave (6.39). However, unlike (6.37), the radius of curvature is not equal to the distance z from the origin: the wavefronts are spheres, but they are not concentric. It is shown in Ex. 6.4 that, in order for (6.43) to satisfy the complex profile equation (6.25), the following relations must hold:

$$w_z = w_0\sqrt{1 + \frac{z^2}{z_0{}^2}} \tag{6.44}$$

$$R_z = z\ (1 + \frac{z_0{}^2}{z^2}) \tag{6.45}$$

$$\eta(z) = \tan^{-1}\frac{z}{z_0} \tag{6.46}$$

where

$$z_0 = k\frac{w_0{}^2}{2} \tag{6.47}$$

Fig. 6.3 illustrated the XZ crossection of a propagating circular Gaussian beam. The 1/e contour line is stylized. The "break point" z_0 indicates where the beam has spread to $\sqrt{2}$ its initial width.

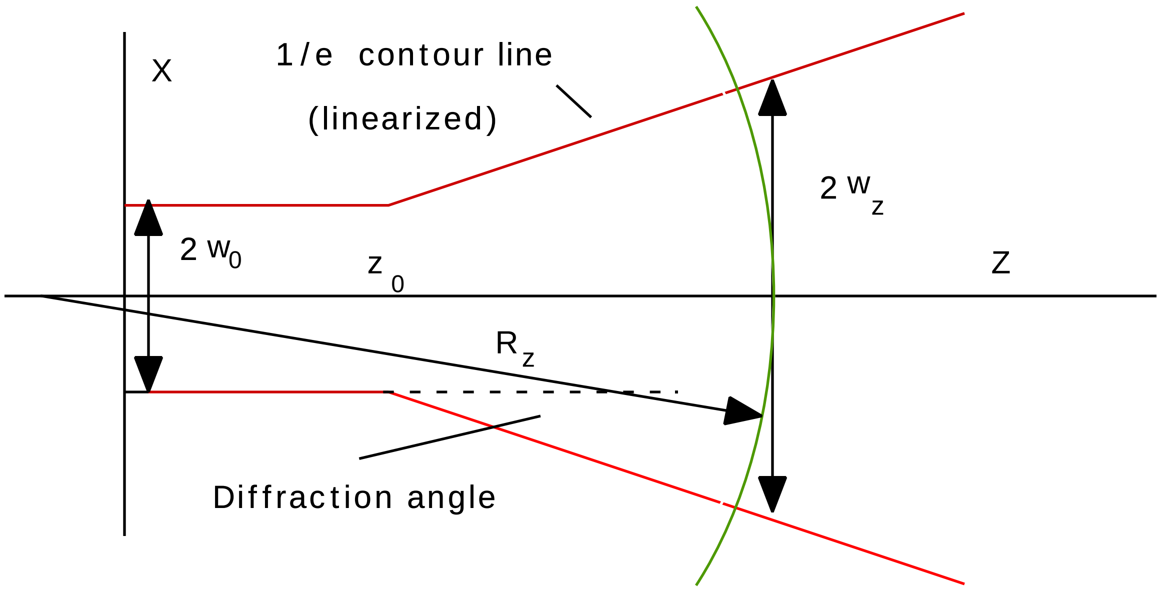

Fig. 6.4

ELLIPTICAL GAUSSIAN BEAM

The field of an elliptical Gaussian beam is given by:

$$E(x,y,z) = E_0 \sqrt{\frac{w_{0x} w_{0y}}{w_{zx} w_{zy}}} \exp(+jkz) \times$$

$$\exp\left[-j\eta_x(z) + j\frac{kx^2}{2R_{zx}} + \frac{x^2}{w_{zx}^2}\right] \exp\left[-j\eta_y(z) + j\frac{ky^2}{2R_{zy}} + \frac{y^2}{w_{zy}^2}\right] \quad (6.48)$$

where w_{0x} and w_{0y} are the waists at the origin in the XZ plane and YZ plane respectively; w_{zx} and w_{zy} are those waists at z; and R_{zx} and R_{zy} are the radii of curvature in the XZ plane and YZ plane at z.

The following relations apply:

$$w_{zxy} = w_{oxy}\sqrt{1 + \frac{z_{xy}{}^2}{z_0{}^2}} \qquad (6.49)$$

$$R_{zxy} = z\,(1 + \frac{z_{0xy}{}^2}{z^2}) \qquad (6.50)$$

$$\eta(z) = \frac{1}{2}\tan^{-1}(\frac{z}{z_{0xy}}) \qquad (6.51)$$

where

$$z_{0xy} = k\frac{w_{0xy}{}^2}{2} \qquad (6.52)$$

and the subscript xy applies to either x or y.

EXAMPLES

EX 6.1

We will first show that , strictly speaking, a plane wave traveling in an direction other than Z, while polarized in the X direction, does not satisfy the Helmholtz equation.

The electric field vector is given by

$$\mathbf{E}(x,y,z) = E_0\mathbf{a_x}\exp(-jk.r) = E_0\mathbf{a_x}\exp(-jk_x x-jk_y y-jk_z z)$$

Hence:

$$.\mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = -jk_x E_0\exp(-jk_x x-jk_y y-jk_z z) \neq 0$$

which completes the proof.

A more general statement is that for .**E** to be zero, **E** must perpendicular to the direction of propagation **k**. This may be shown as follows.Let

$$\mathbf{E}(x,y,z) = \mathbf{E_0}\exp(-j\mathbf{k.r}) =$$

$$(E_{0x}\mathbf{a_x} + E_{0y}\mathbf{a_y} + E_{0z}\mathbf{a_z})\exp(-jk_x x-jk_y y-jk_z z)$$

Hence

$$.\mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = -j(k_x E_{0x} + k_y E_{0y} + k_z E_{0z})\exp(-j\mathbf{k.r})$$

This can only equal zero if $(k_x E_{0x} + k_y E_{0y} + k_z E_{0z}) = 0$ and hence **k** must be perpendicular to $\mathbf{E_0}$

Ex 6.2

The spherical wave $E(x,y,z) = \frac{1}{r}\exp(\pm jkr)$ satisfies the Helmholtz equation $^2E + k^2E = 0$.

It is more convenient to recast the Helmholtz equation in spherical coordinates. Because E(x,y,z) only depends on r, we use the following expression for the Laplacian [Ref 2]

$$^2E = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial E}{\partial r}\right) \qquad (6.2.1)$$

After some algebra, we find that

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial E}{\partial r}\right) = -k^2E$$

so that the Helmholtz equation is satisfied.

Ex. 6.3

We will show that the cylindrical wave of 6.21 satisfies the Helmholtz equation $^2E + k^2E = 0$ if $kr >> 1$

We use a cylindrical coordinate system with the axes along Y. Taking into account that E only depends on $r = \sqrt{x^2+z^2}$, the Helmholtz equation may be written as

$$\frac{1}{2k}\frac{1}{r}\frac{\partial E}{\partial r}\left(r\frac{\partial E}{\partial r}\right) + k^2E=0 \qquad (6.3.1)$$

Substituting

$$E(x,y,z) = \frac{1}{\sqrt{r}}\exp(\pm jkr)$$

into (6.3.2), we find

$$-k^2E\left[1-\frac{1}{4k^2r^2}\right] + k^2E = 0 \qquad (6.3.2)$$

We see that (6.3.2) is satisfied if $kr>>1$.

Ex. 6.4

Propagation of a circular Gaussian beam.

We assume that the beam is given by

$$E(x,y,z) = A(z)\exp[-a(z)(x^2+y^2)]\exp(jkz) \qquad (6.4.1)$$

so that the complex profile E_e is

$$E_e = A(z)\exp[-a(z)(x^2+y^2)] \qquad (6.4.2)$$

Applying the complex profile equation

$$\frac{\partial^2 E_e}{\partial x^2} + \frac{\partial^2 E_e}{\partial y^2} + 2jk\frac{\partial E_e}{\partial z} = 0 \qquad (6.4.3)$$

to (6.4.2) we find:

$$-4a + 4a^2x^2 + 4a^2y^2 + 2jk\frac{1}{A}\frac{dA}{dz} - 2jkx^2\frac{da}{dz} - 2jky^2\frac{da}{dz} = 0$$

$$(6.4.4)$$

Setting the constant terms and the coefficients of x^2, y^2 separately equal to zero, we obtain:

$$-2a + jk\frac{1}{A}\frac{dA}{dz} = 0 \qquad (6.4.5)$$

$$2a^2 - jk\frac{da}{dz} = 0 \qquad (6.4.6)$$

From (6.4.6) it follows immediately that

$$a(z) = \frac{1}{w_0^2 + 2jz/k} \qquad (6.4.7)$$

where $w_0^2 = a(z=0)$, the beam radius at the origin.

Eq (6.4.7) may be written as:

$$a(z) = \frac{1}{w_0^2(1 + \frac{4z^2}{k^2 w_0^4})} - \frac{jk}{2z(1+\frac{k^2 w_0^4}{4z^2})} \qquad (6.4.8)$$

Defining

$$z_0 = kw_0^2/2 = \pi w_0^2/\lambda \qquad (6.4.9)$$

we may write

$$a(z) = \frac{1}{w^2(z)} - \frac{jk}{2R(z)} \qquad (6.4.10)$$

where

$$w^2(z) = w_0^2(1 + \frac{z^2}{z_0^2}) \qquad (6.4.11)$$

$$R(z) = z(1+\frac{z_0^2}{z^2}) \qquad (6.4.12)$$

Combining (6.4.5) and (6.4.6) gives

$$\frac{1}{A}\frac{dA}{dz} = \frac{1}{a}\frac{da}{dz} \qquad (6.4.12)$$

whence A = Ca

At z=0, $A=A_0$ and $a = \frac{1}{w_0^2}$ hence $C = A_0 w_0^2$

With (6.4.7) it then follows that

$$A(z) = \frac{A_0 w_0^2}{w_0^2 + 2jz/k} = \frac{A_0 w_0^2}{\sqrt{w_0^4 + 4z^2/k^2}} \exp[-j\tan^{-1}(\frac{2z}{kw_0^2})]$$

`` (6.4.13)

With (6.4.9) and (6.4.11) this may be written as

$$A(z) = A_0 \frac{w_0}{w(z)} \exp(-j\eta(z)) \qquad (6.4.14)$$

where

$$\eta(z) = \tan^{-1}(\frac{z}{z_0}) \qquad (6.4.15)$$

REFERENCES

1. S. Haykin, *Communication Systems* , 3rd ed., Wiley, New York, 1994
2. D. K. Cheng, *Field and Wave Electromagnetics,* 2nd ed., Addison-Wesley, New York, 1989.